The conclusion for educational strategies is not referentiality of (formal) language emerging from autopoietic processes of individual learning.

fluent, but gets along without explicit application of underlying laws. The inability in applying laws is formal mathematical language as belonging to the world of natural language: usage is correct and interpreted. Data show a lack of basic competences and essential insight in mathematics.

Analysis of the data is approached by psycholinguistics: language is considered the primary carrier of human thought and consciousness. Three worlds of language are distinguished: natural language, scientific language and formalised language. Self-referentiality distinguishes the tree worlds: natural language is self-referential by its very nature, scientific language tries to exclude self-referentiality in favour of hierarchically structured theories ideally based on natural constants, and in formalized language of mathematics self-referentiality pops up again on an abstract level.

Interpretation following the psycholinguistic approach is: the data show usage and understanding of formal mathematical language as belonging to the world of natural language: usage is correct and fluent, but gets along without explicit application of underlying laws. The inability in applying laws is not interpreted as lack of specific exercises in respective curricula, but as lack of familiarity with self-referentiality of (formal) language emerging from autopoietic processes of individual learning.

The conclusion for educational strategies is not to ramp up mandatory science education in traditional kinds of classes, but to systematically find individual talents and further them specifically.

Language in Science Education – a Psycholinguistic Approach

Abstract

Empirical data from a questionnaire with science students from five countries are presented and interpreted. Data show a lack of basic competences and essential insight in mathematics.

Analysis of the data is approached by psycholinguistics: language is considered the primary carrier of human thought and consciousness. Three worlds of language are distinguished: natural language, scientific language and formalised language. Self-referentiality distinguishes the tree worlds: natural language is self-referential by its very nature, scientific language tries to exclude self-referentiality in favour of hierarchically structured theories ideally based on natural constants, and in formalized language of mathematics self-referentiality pops up again on an abstract level.

Interpretation following the psycholinguistic approach is: the data show usage and understanding of formal mathematical language as belonging to the world of natural language: usage is correct and fluent, but gets along without explicit application of underlying laws. The inability in applying laws is not interpreted as lack of specific exercises in respective curricula, but as lack of familiarity with self-referentiality of (formal) language emerging from autopoietic processes of individual learning.

The conclusion for educational strategies is not to ramp up mandatory science education in traditional kinds of classes, but to systematically find individual talents and further them specifically.
1. Introduction

“In recent years, many studies have highlighted an alarming decline in young people's interest for key science studies and mathematics. Despite the numerous projects and actions that are being implemented to reverse this trend, the signs of improvement are still modest. Unless more effective action is taken, Europe's longer term capacity to innovate, and the quality of its research will also decline. Furthermore, among the population in general, the acquisition of skills that are becoming essential in all walks of life, in a society increasingly dependent on the use of knowledge, is also under increasing threat.” – these are the introductory words in the executive summary of a recent report of the High Level Group on Science Education of the European Commission [EU07].

Addressing the above challenges, an R&D group on mathematics software run into discussions about the principal role of software in mathematics. The discussions led to an EU-wide questionnaire; the data collected by the questionnaire are presented in this paper. The attempts to analyze the data, however, led to even more heated discussions. Finally Yuri Manin's remark “mathematics is a linguistic activity” [Manin07, p.34] motivated the group to include linguistics into the scope of discussion. This paper presents the outcome of the interdisciplinary analysis of the questionnaire's data.

The interdisciplinary approach found the most appropriate common ground in the emerging theory of complex systems in general and in psycholinguistics in particular. Semiotics [Manin07,p.186] regards information processing as the proto-notion of complex systems. In the sequel the word “complex” consistently refers to this specific kind of complexity.

The human central nervous system (CNS) as complex system can only be understood in human social context (another complex system) if mutual development of CNS, thought and language is under consideration: Society consists of individuals, and a human individual cannot become human without social context as we know from “wolf children”. This raises the interesting question, how consciousness and language came into existence, if there was no “human” social context at the beginning of transition from pre-apes to pre-humans. This question is addressed in §2.1.

Human language refers to the structure of control and informational processes in both systems, the individual CNS as well as in society [Manin07,p.71 pt.5]. So, when considering science education we address the phylogeny of science and of language as well as ontogeny of language in science education of the individual. Both developments will be separated into levels by time and by characteristics; the levels range from natural language to formal language of mathematics. Proceeding from level to level in phylogeny and in ontogeny is considered inevitable.

Equipped with that theoretic background the above mentioned discussions cleared up and the subsequent analysis of the questionnaire's data became possible.

The paper is organized as follows: §2 starts with a review of psycholinguistics; after recalling known facts about origins of thought and language in §2.1 three stages of development are identified from the phylogenetic point of view in §2.2 and from the point of individual maturing and education in §2.3. The stages are separated into natural language (§2.2.1 for phylogeny and §2.3.1 for individuals), scientific language (§2.2.2 and §2.3.2) and formal language of mathematics (§2.2.3 and §2.3.3). From this theoretic background the empirical study is presented: §3.1 presents the methodology, §3.2 presents the data and §3.3 gives an analysis and an interpretation. The study gives raise to recommendations for didactics in §4 and §5 gives a summary and a preview to future work.
2. Reviewing Neuropsychology of Language and Thought

Philosophy, neurology, pedagogy, all clearly distinguish between the notions of “language” and the notion of “thought”. For instance, George Polya [Polya57], in unison with many other mathematicians and physicists, clearly states that mathematical thinking consists of parts not covered by language.

In contrary, this paper closely relates these two notions and almost takes “thought” and “language” as synonyms. One argument for doing so is: Science is a result of human thought, however, what is captured by language finally is the only part which can be communicated in the present state of science. Communicating knowledge is one of the indispensable goals of science. So in the paper’s approach language is the main road into discussion of thought and language. Other reasons are put into perspective in the subsequent section.

2.1. On Origins of Thought and Language

The mystery of human thought and language and respective origins has been tackled by various scientific research: “Comparative linguistics has made possible deep reconstructions of ancient languages […] going back to the preliterate period, possibly ten thousand years ago. It includes […] research on animal behaviour and human psychopathology, data on the development of syntactic constitution of the modern languages and reports on effects of hallucinogens; interpretation of myths and theory of databases; finally, resort to introspection and psychological gedanken experiment” [Manin07, p.170].

Language emerged in co-evolution with the brain. The volume of the brain developed from 350ccm 3.2 Mio. years ago (“Lucy”) to 700ccm 2 Mio. years ago (homo rudolfensis) to finally 1.360ccm 100.000 years ago, the size of an average adult’s brain at the present. Beyond size, the feature most distinctive from a primate’s brain is lateralisation.

Lateralisation has emerged in co-evolution with language, since the differences concern speech centres located in the left hemisphere (for most right-handers). Julian Jaynes [Jaynes76] discusses dramatic consequences of a “bicameral mind”, some of which have been confirmed by sophisticatedly designed tests on patients with hemispheres separated by surgery in order to relief epilepsy: the two hemispheres expressed different emotions and intentions (while only the left side became conscious such that the testee could report). Such phenomena are still known from certain kinds of mental illness; in medieval times hearing voices (from the other hemisphere [Jaynes76]) was still acceptable, remember Jeanne d’Arc.

One needs not follow Jaynes in all details to acknowledge, that the emergence of lateralisation was accompanied with highly unreliable and unstable mental states of prehistoric humans, who have lost instinctive guidance and who were far from the state we addict to a present human. In those times the role of advanced persons, who somehow could master the neurophysiological challenges raised by lateralisation, was even more decisive than today: their leadership made their tribes fit for survival in evolution. Manin calls such persons tricksters: The trickster is a memory about humanity “through which the language learns to speak”; through which for the first time strange capabilities for speech become apparent: the violence of taboos and lies to circumvent them which starts hearing the speech of everything around it: trees, water, rocks, its own body and the gods — and tries to decipher this language; whose “agony of words” is a fact a physical agony, which we barely remember after many generations. [Manin07, p.183].

In this light language is the mental mechanism which creates the temporal-spatial-social continuum in the present human mind. Story telling and counting structure time in our minds (Robinson Crusoe is told to have used a calendar in order to maintain his humanity); spatial capabilities of animals (of migratory birds, for instance) are stunning, but we measure space in various ways; and last not least the
development of present society is unthinkable without language. Thus Humberto Maturana [Maturana88] calls human existence “living in language”.

### 2.2. Phylogeny of Language and Science

The previous section showed the dramatic and (in terms of evolution) short development of human mind and clarified the role of language as a nucleus and as a carrier of human thought.

For the purpose of this paper the continuous, but not linear development of language is divided into three stages, where the order of the stages expresses implication: each stage is an indispensable prerequisite for the subsequent stages. The social dimension of thought is reflected by science, so science is discussed in parallel within the development of language.

#### 2.2.1. Phylogeny of Natural Language

TODO from the point of view of linguistics by respective expert(s)

Probably required by other parts of the paper: Language addresses objects and other humans, so addressing the self is a straightforward continuation of verbal communication. Watching verbal communication reveals, that the majority of words are related to other words (spoken by the partner), not related to immediate action. So, self-referentiality is inherent to the nature of language and self-perception is unavoidable.

A particular statement referred to in §4 of the paper: natural language is sufficient for human development and human existence exemplary until today – remember Jesus, Mohammad, Buddha, Laozi; all these founders of a religion did not require later stages of scientific language or even more formal language. This statement will be used to ask, whether these later stages are amenable to exaggeration which might endanger balanced human existence.

<table>
<thead>
<tr>
<th>Natural language in brief:</th>
</tr>
</thead>
<tbody>
<tr>
<td>serves immediate personal communication and is amenable to self-referential use</td>
</tr>
</tbody>
</table>

#### 2.2.2. Phylogeny of Scientific Language

Science is already found in the eldest cultures leaving written witness. Evident prerequisites for developing science and respective language are

- a sufficient high percentage of individuals in a society “living in language” [Maturana88]
- a sufficient large portion of written language on myths, religion, arts etc
- a sufficient large number of privileged people set free from primary production and spending effort on thinking beyond the needs of the day.

Several coincidences were necessary for establishing these prerequisites. In privileged areas the 'neolithic revolution' introduced agriculture producing a surplus of food. First cities developed in Mesopotamia, administration required written records and numbers; in Egypt regular floods of the Nil required re-establishing acres and thus geometry.

Modern science arose from an idea, looking simple nowadays, but by no means obvious in the initial period during Renaissance; one remembers Galileo Galilei and the discussion of his methods by [Feierabend.TODO]. The idea is: verify statements by relating it to “reality” in a systematic way.

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1 At least, science in the modern sense was not required. However, all the founders of a religion had considerable bodies of written knowledge at their disposal.
Reality is quoted in order to stress, that scientific methodology tries to exclude the societal and the self-referential aspect of knowledge. That is one of science's strengths: experiments can be repeated by anybody everywhere at any time.

In these decades a change of paradigm can be observed from a “science based on classical mathematics” to a “science of complex systems”. Some of physics and of biology has already turned to acknowledge the surprising properties of complex systems, while the rest of science still relies on abstraction as supported by classical mathematics (without emerging theories on complex systems). An typical example is behaviorism: for the sake of verifiability systems are simplified by dropping all internals of human beings. Many enterprises and institutions are still organized according to that paradigm, in particular educational systems: curricula are being standardized, exams are being centralized etc.

Scientific language and thought in brief:
clarifies statements by elaborated agreement, statements are verified by relation to “reality”

2.2.3. Phylogeny of Formal Language of Mathematics

Aristotle was the last scientist who found natural language sufficient for his scientific work [Manin07, p.TODO] – driven by development of natural sciences, the objects of study and the constructs of thought became too complicated for appropriate description in natural language. Already Diophantus of Alexandria developed a semi-verbal notation for his mathematical discussions. In Renaissance Viete and others developed an algebraic notation, which was much more comprehensible and manageable [Manin07, p.TODO]. From then “science relegated natural language to the role of a mediator between knowledge and human brain [Manin07, p.212].

Now, what about verification as demanded by scientific methodology? Or more naively: Why mathematical statements should be taken serious, if not verified against reality? Verification against reality is replaced by verification in terms of logic. “Logic teaches us that certain formal constructions produce truthful statements when applied to truthful statements. Mathematics uses such constructions recursively. All comparisons with reality is relegated to comparatively scarce encounters with applications and, possibly, foundational studies. The main body of mathematical knowledge looks like a vast mental game with strict rules” [Manin07, p.33].

Such rules determine math argumentation called a proof. So, since Euclid a proof consists of exactly defined terms (which are denoted by formulas today) combined by arguments in natural language due to logics. The development of mathematical proof more and more restricted the scope of natural language and extended the scope of formulas. For instance, the equality sign “=” became part of formulas about 1550. Since then algebra is written as

\[ 2.c + 3.c + 4.d + 5.d = (2 + 3).c + (4 + 5).d = 5.c + 9.d \]

Argumentation also can use formulas: the first of the above equality signs is justified by the law of distributivity, \( a.c + b.c = (a + b).c \). Such application of formulas justifying manipulation of formulas is the basement of formal argumentation, the fundament of mathematic proof at the core of scientific methodology as we agree upon in present science.

With the advancement of mathematics the objects of study became more and more complicated, for instance, infinity needed reliable treatment. Scholastic and propositional logic were developed further to predicate calculus and other logics like first order logic, high order logic, temporal logic, etc. The high time in developments of logics was about a century ago, when problems were encountered (for
instance, David Russel's “set of all sets which contain themselves”) which called the foundations of mathematics into question. The phase of clarification of mathematic’s foundations led to the milestones set by Kurt Gödel, who formalized even self-referentiality of certain formal languages.

The second half of the last century advanced the formalization of proof to the mechanization of proof by computer theorem provers. Such machines cannot invent proofs – this becomes clear by a look to the famous book of George Pólya [Polya57]. However, computer theorem provers reliably check proofs for correctness by breaking down a proof to elementary steps beyond doubt. The drawback of a proof broken down into elementary steps is incomprehensibility due to the large number of steps. Recently even this problem is being tackled successful by mechanized provers 2.

With respect to the features of language discussed in §2.2.1 it is important to state, that finally the self-referential nature of language pops up again in the formal language of mathematics and appears in the rigorous clarity typical for mathematics.

**Formal language in brief:**

- statements are verified by abstract rules expressed by formal language in self-referential way

### 2.3. Ontogeny of Language and Education

So far it has become clear, how learning is considered a complex autopoietic process, which can hardly be predefined. However, humans like planning, and so do they in the development of the individuals in the next generation. In this respect it needs to be clear, that “information transfer” as, for instance, realized by computers, is no appropriate model for learning and that predefined lessons tend to distract from learning as a complex process.

The three stages from the discussion of phylogeny will be reused for individual development; with respect to individuals should be clear: “The functioning of a person at a given age may be so variable from domain to domain, such as the understanding of social, mathematical, and spatial concepts, that it is not possible to place the person in a single stage” 3.

Self-referentiality will be in the focus, since it will be referred to by the interpretation of the empirical study in §3.3.

#### 2.3.1. Education and Natural Language

TODO from the point of view of linguistics by respective expert(s)

Probably required by other parts of the paper: Natural language builds upon predispositions selected by evolution (mirror neurons, speech centers in the left hemisphere); the autopoietic process of learning can be described simply by imitation, trial and error(?); such learning needs not include consciousness about underlying structure of language (syntax, grammar).

Example for learning processes which involve self-referentiality: *irony* (TODO by expert)

#### 2.3.2. Education and Scientific Language

TODO from the point of view of linguistics by respective expert(s)

Probably required by other parts of the paper: requires written language; development and usage of technical terms in scientific language requires self-referential use of language and thus self-perception.

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Example for learning processes which involve self-referentiality: Adoption of technical terms by immigrants: they adopt natural language in communication with their school mates, but there are difficulties in adopting technical terms. Possible interpretations: these technical terms require certain experiences not provided by the cultural background of the immigrants (for instance, the term “human rights” require the experience, that one is allowed to express personal opinions different from others, that different habits are just different and can be maintained due to personal agreement, etc).

2.3.3. Education and Formal Language of Mathematics

One generation ago in Europe formal aspects of language were taught in a way, which seems to be considered a detour by current curricula: “Personally I learned thinking best by instruction in classical Greek. One read a complicated sentence from Plato, gathered vocabulary and complex grammar and then started thinking. If one has recognized the structure, one started to puzzle over while asking, whether the translation is meaningful. Ancient Greek is complicated such that one has to reorganize frequently in order to make a combination meaningful. And exactly that could mathematics accomplish, too.” [Kreck10, p.87, translation from German].

The above words “complex grammar” and “structure” (emphasised by the authors of the paper) refer to a kind of “lexical analysis”, where the particular letters of the endings play the essential role in a first step – the endings must be matched with several possible memberships in declinations and conjugations. Semantics comes in a second step, if some structure has been detected – the dictionary will be consulted. And then the two kinds of steps are iterated, called “puzzle over” above.

By this way instruction in ancient languages was an efficient training to become familiar with formal aspects of language (endings, but no meaning). Such familiarity is required in mathematics as well, when matters become complicated such that natural language cannot describe them sufficiently anymore: if there are some symbols denoting something not yet understood, it may be useless to relate the letters to imagination of some “real” thing; rather one has to search for definitions. Such definitions consist of symbols again, but these should represent more elementary notions, probably reachable by intuition. Examples help to gain “some imagination” at this point.

This point is comparable with a recent discussion on legislative texts: Laymen asked to translate the 200 years old ABGB to present German. The answer of judicial experts was: Translation would hardly improve comprehensibility, because many old terms had been defined in a centralized effort; translating these terms would lead to modern terms with connotations different from those carefully reflected in the original effort; so translation would change definitions and defining terms which would end up to redo the whole ABGB.

The analogy to mathematics is: the terms have a relation to experience (which is necessary for comprehension), but the meaning determined by definitions is the relevant one. The more complex the concern of the text is, the more the definitions become relevant, and the less “common sense” is reliable.

Looking at such analogies seems to reveal, that the difficulties for learners show up at the point, where language must be used in a self-referential way, where no relation to “reality” and no direct intuition (i.e. approval from the right hemisphere in the brain) is possible. Rather, at this point formal operations (looking up definitions) are required. Mathematics goes beyond legislative texts in that the atomic definitions do not refer to some reality (to “natural law”, for instance), but remain just formulas (of formal logic).

This intricacy of self-referential use of language (and of thinking) is usually not conscious to the experts in this kinds of thinking. In the same interview cited above, Matthias Kreck [Kreck10, p.87, translation from German] says: “At school education mathematics is being reduced to proficiency in
formula manipulation. This does not meet the nature of mathematics.” Right, if students learn formula manipulation just like a natural language (without further insight), this is only a part of the whole nature of math – but if dropped, this indispensable part is missing as stated in [Kirsch06, Sweller85, Jeroen07].

TODO: bring the two parts of math, natural and formal language, together in terms of linguistics.

3. Empirical Study on Language in Mathematics

The study concerns that part of mathematical language which is not only the eldest one, but also the best known and most exercised, the language of algebra. In all curricula algebraic notation is introduced at an age about twelve and appears indispensable throughout high school math and in math applied in science at universities. The most basic notion in this language is “algebraic law”, for instance, “(a + b).c = a.c + b.c” is the “law of distributivity” (because it distributes the factor c over the elements a and b). The study addresses this basic concept and respective usage.

3.1. Methodology of the Study

The study was performed by a questionnaire filled by 300 students in engineering studies from five universities in four countries, Austria, Romania, Serbia and Spain between February and May 2010. The questionnaires were filled by the students in 30 minutes during regular lectures; the verbal introduction just repeated the text written at the beginning of the questionnaire; no help was given.

The questionnaire is double-sided. The first page addresses two different aspects of the concept “algebraic law”. Question (1) checks, if the concept is known and understood: (i) the formal pattern, (ii) the name and (iii) some understanding (a law is valid for many individuals). An example (a) is followed by items (b) ..(g).

1) Below you find basic laws of algebra (i.e theorems). (i) **Do you remember some laws?** (ii) Do you even remember the names of some laws? (iii) Can you apply the laws to numbers?

   a) a + b = b + a 
   (iii) 2 + 3 = 3 + 2 … 5 = 5  
   b) a · b = b · a 
   (iii) = = = = …
   c) (a + b) + c = a + (b + c) 
   (iii) = = = = = = = …
   d) a·(b + c) = a·b + a·c 

Question (2) checks, if algebraic formula manipulation is fluent without asking for explicit application of algebraic laws. An example (a) is followed by items (b) ..(e).

2) **Simplify the following algebraic expressions, please.** Simplifying such expressions is learned together with laws of algebra; but usually one simplifies without laws, for instance:

   a) 2(x + 3·y) – 6y = 2·x + 6·y – 6·y = 2·x .
   b) 2(x + 3·y) + 6y = ………………………………………… =
   c) r + r(2 + s) = ………………………………………… =
   d) (u + 1) · (u – 1) = ………………………………………… =
(x + y) · (x – y) = ................................................................. =

On the second page question (3) checks, if algebraic laws can be applied to exactly those examples which have been checked in the previous question without relation to laws. An example is followed by items (a)–(c) and a similar question not shown here.

This page is about using laws to justify steps in simplifications. We give the following abbreviations for laws:

[C+] a + b = b + a
[A+] (a + b) + c = a + (b + c)
[A–] (a + b) – c = a + (b – c)
[U+] a + 0 = a
[U·] a · 1 = a
[I+] a – a = 0
[D+] a·(b + c) = a·b + a·c

Here is an example of stepwise justifying a simplification by use of these laws and by calculating natural numbers [N+–·]:

\[ 2 \cdot (x + 3 \cdot y) – 6 \cdot y = \] 
\[ = [(A+)] (2 \cdot x + 2 \cdot (3 \cdot y)) – 6 \cdot y = \] 
\[ = [(A–)] 2 \cdot x + (6 \cdot y – 6 \cdot y) = \] 
\[ = [I+] 2 \cdot x + 0 = \] 
\[ = [U+] 2 \cdot x . \]

3) Similarly describe a stepwise justification of the following simplifications, please;
Take as many steps you need:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = \] 
\[ = [(…..)] \] 
\[ = [(…..)] \] 
\[ = [(…..)] \] 

b) ...

The questionnaires were given in the respective first (mother) language, German in Austria (but there were some foreign students in Linz), Serbian in Belgrade and Spanish in Madrid. The exception were English questionnaires in Romania.

TODO… comparative study with high school students

TODO… comparative study on lack of time for completion or on decrease of attention due to the long time of thirty minutes to fill the questionnaire …

3.2. Presentation of the Empirical Data

The numbers below are projections drawn by eyes on a selection of questionnaires; TODO detailed counting.

3.2.1. A Firm Concept of Algebraic Law

The answers to question show, that students are familiar with the concept of “algebraic law”:

i. A great majority of students have seen most of the laws, probably during math education.

ii. Names of the laws are not so well known, except in classes with math students

iii. The essential insight is present with almost no exception, that a law stands for arbitrary individuals (in this case for constant numbers).
3.2.2. Correct Manipulation of Algebraic Formulas

The answers to question (3) show, that a great majority of students simplifies all algebraic terms correctly:

![Chart showing correct and incorrect responses]

3.2.3. Minimal Use of Laws in Manipulation

Question (3) asks for using given laws, shown to be well understood in question (1), for justifying steps in simplifications, shown to be well mastered in question (2). In contrary to question (1) and (2), this question shows opposite results: students show very little success in applying the laws in simplification; the majority has not idea (“0 correct” below).

![Chart showing correct and incorrect responses]
3.3. Interpretation and Analysis

3.3.1. Algebra is learned and used like a natural language

The empirical data above show algebra used like a natural language by a vast majority of students: they manipulate formulas correctly, reliably and fluently (question 2) without consciousness about underlying laws and rules (question 3). The interesting point is, that the same majority has a firm concept of the notion of “algebraic law” (question 1).

A quick interpretation for this phenomenon was: the concept of “algebraic law” is known but not yet applied – as is the case in general with the acquisition of competences. If this interpretation would apply, then the comparative study with high school students would show weaker results for high school and better results for university; this is not the case. At this point our interdisciplinary approach comes to bear; this approach illuminates already certain points in algebra curricula at schools in a novel way.

An investigation of algebra curricula by reviews of math textbooks for high schools shows, that the concept of “algebraic law” is introduced at the age of about twelve years. There considerable effort is spent for explanations relating the laws to images and other experiences of reality, for instance the “law of distributivity”:

![Fig.1 Explanation for the law of distributivity, \((a + b).c = a.c + b.c\)](image)

In parallel to the introduction of laws the usage of the laws is shown and exercises in the textbooks; most usage concerns simplification of formulas, since this is most motivating (question (2) concerns such simplifications). Interestingly, in almost all textbooks the first step of simplification is described verbally. For instance, given a formula like \(2.c + 3.c + 4.d + 5.d\) this step is described verbally as
“collect the terms having the same variable” and “you cannot add citrons like \( c \) to deers like \( d \)”, and the description does not use the already introduced law of distributivity (read from right to left):

\[
2c + 3c = (2 + 3)c = 5c
\]

Also in all the other steps in algebraic simplification, rigorous application of laws is not suggested by the exercises. This is well acceptable, since such application would slow down simplification, while fluency in algebra increases the usability of this important and basic competence.

A remarkable large percentage of teachers, pointed at the missing relation between laws and respective application in simplifying already, pledged for adapting the textbooks to rigorous application of laws to simplification. Advocating the knowledge from psycholinguistics as introduced in §2, however, gives good reasons, that the designers of the textbooks are right in the kind of introduction to algebra they decided for – without strong relation between laws and respective application. These reasons are concern of the subsequent section.

### 3.3.2. Self-Referentiality of Formulas Involves Complex Thinking

Reformulating the concern from above, rigorously applying laws to algebraic simplification, with respect to the role of proof in mathematics as discussed in §2.2.3, would be

*justify the manipulation of formulas by another kind of formulas*

where “justify ...” even could be replaced by “prove the correctness of ...”. And reformulated again with respect to the knowledge from psycholinguistics as introduced in §2, would be

*consciously use the self-referential nature of language in the formal language in mathematics*

This kind of request clearly addresses the same level of complexity as discussed for irony in §2.3.2 and requires a similar familiarity with self-referentiality of language. Learning in such context is considered an autopoietic process in accordance to neurophysiological maturing of the brain. The neurolinguistic point of view cannot recommend to change math textbooks such that rigorous application of laws is introduced together with algebraic simplification. Data from §2.3.2 on problems of immigrants learning scientific language clearly indicate, that certain competences involve complexity such that simple “teaching and learning” in prepared lessons does not help.

An extreme analogy might clarify this point for mathematicians: teaching students calculate so-called “Gödel numberings” would not be more difficult than teaching them rigorous application of laws in algebraic simplification (this should be proven experimentally !) – but would such teaching promote understanding of Gödel's so-called “incompleteness theorem”? Gödel numberings are the technique formalizing self-referentiality of language in this theorem, which leads to consequences significant for mathematics in general such that, for instance, David Hilbert, the most prominent colleague, never adopted this theorem; otherwise the words “Wir müssen wissen. Wir werden wissen.” probably would not be written on Hilbert's gravestone.

From the point of neurolinguistics a planned sequence of math lessons with the topic “rigorous application of laws in algebraic simplification” cannot succeed in traditional school classes; rather it would establish a shallow kind of competence covering the fact, that for most students there is no deep insight to the self-referential use of formulas in particular and no insight in self-referentiality of language in general.

Interestingly, verbal discussions about the questionnaire indicates, that the majority of lecturers at universities of technology expect their students to have the competence of “justifying algebraic
simplification by rigorous application of laws”. Some lecturers add: “Otherwise the students could not pass the exam at the end of the lecture”.

4. Conclusions for Didactics

The previous section presented as fact, that there are competences which are hardly appropriate for traditional schooling – but which are considered indispensable requirements for science studies by lecturers at universities of technology. In principle, there are at least two ways to close the gap between requirements and the actual competences:

1. **Ramp up the efforts in science education in order to reach the goals**, in particular intensify the competences identified as weak. This is the way proclaimed presently: push more and more young people into science education according to OECD goals; in particular, make teacher education academic even for personnel in kindergarten.

   This way seems not to be separable from the deficiencies recognised in traditional schools: pupils have more answers than questions and don’t know what they are interested in after twelve years of sitting in school. Of course, these effects are consequences of the learning methods and up to improvement; but probably to a lesser degree than acknowledged today: the best methods do not help, if the content of learning is inappropriate. In complex learning the appropriateness of content is determined by the learner.

   So, this way is not considered promising from the point of view taken in this paper.

2. **Reconsider the role and the goals of science and of education** according to the stages of thought and language as presented in §2: Stage two of scientific language (and even more stage three of formal mathematics, see §4.2 below) is considered a spin-off from the first stage called “natural language”. This first stage may considered the base case providing all prerequisites for full development of human personality (with founders of religions as models).

   Pushing science education on young people not ready for that offer diminishes their capacities for developing their very personal strengths. The data presented in §3 suggest, that there are many young people misguided to dissipate their strengths by education.

   There is the arguments “But we need more scientists!” Our answer is “Yes, but not scientists just capable of copying and learning by heart.” What we need is a distribution of labor from handcraft to science which meets the distribution of interests and talents in the population (and those talents are at least partly related to the stages described in §2).

   These considerations are general and for the long term, of course. More specific and more short term conclusions are given subsequently.

4.1. Didactics of Language Instruction

TODO

4.2. Didactics of Mathematics

§3.3.2 stated, that rigorous application of laws in algebraic simplification cannot be simply taught to a school class and learned in a planned sequence of lessons – on the other hand §2.2.3 stated, that such application is an indispensable prerequisite for formal argumentation and thus an indispensable prerequisite for science education successful on academic level. How can didactics cope with this conflict? This short paper gives some remarks on future research, some hints for teachers in the first paragraph and for administrators in the second paragraph below. Both hints may take “justification of algebraic simplification” as an example for complex competences.
Acknowledge, that complex competences can not be simply “taught and learned”, not in a well prepared lesson by a genius of a teacher supported by perfect media, and not even in a sequence of such lessons. Rather, be aware

- that intensified exercising does not increase complex math competences, if the students' brain is not ready to accept the stimulus, but creates shallow pseudo-knowledge and pseudo-competences, see Matthias Kreck's complaint that “at school education mathematics is being reduced to proficiency in formula manipulation. This does not meet the nature of mathematics” [Kreck10] cited in §2.3.2. Frequently such a pseudo-competence evokes “I already have learned that” from a student facing a matter once again – probably at a time, where his or her mental maturity would actually be ready for new insights.

- that complex competences emerge suddenly, probably after some latency period where several mental prerequisites established in the autopoietic process of learning. Research has to isolate these prerequisites in order to clarify the role of educational planning. This paper stresses self-referentiality in natural language as a prerequisite for “justification of algebraic simplification” in formal language. Most likely there are other prerequisites as important.

Slow thinkers are often the better thinkers, as biographies of acknowledged scientists show – these are definitely discriminated against overachievers in present schooling.

- that stages of mental maturing cannot be accelerated (and only stimulated), and complex competences dependent on certain maturity of the brain cannot be accelerated as well. There are lots of theories about developmental stages, and only few of them have been applied to mathematics education. The allocation of self-referentiality in the different developmental stage theories is not settled, too.

Organize learning around a variety of offers to individuals as much as possible, keep obligatory classes in mathematics at a minimum – as such classes are, they seem to do more harm to science education than is acknowledged presently. Rather

- organize learning scenarios by open questions, inquiry, trial and error, by collaborative work, communicating questions and variants of solutions by social activities, etc. The new curricula, for instance in Austria, go into the right direction – however, it must be clear, that these innovations shift attention away from activities important as prerequisite for science studies (for instance, “rigorous justification of algebraic simplification”). [Kirsch06, Sweller85, Jeroen07] clearly state, that “teaching general problem solving does not lead to mathematical skills or knowledge”.

- differentiate the organization of classes at schools; increase the number of elective courses, in particular, offer specific courses as mandatory preparation for science studies (as done in Finland, for instance); improve mechanisms identifying specifically motivated and gifted individuals; support discussion circles in mathematics, which were so successful in Eastern Germany before the breakdown of Communism; support all kinds of individual and independent learning.

It can be expected, that the loss in the number of students in mandatory math classes at high schools can be balanced by a gain of students who not any more are discouraged by the inescapable features of mandatory classes for the masses.

- improve mathematics assistants to software which model a large portion of mathematics: such software can actually be designed as models of mathematics [Neuper10]: interactive solution of interesting examples could trace justifications down to elementary justifications. By this way such assistants could provide a permanent offer to investigate the different levels of language underlying self-referentiality, to investigate related abstract knowledge of mathematics while allowing relief from the intricate details of formal operations at the same time.

Computers are a straight forward development of the “linguistic activity” mathematics: they are symbol
manipulating machines where the “symbols” are managed by formal languages in several levels implementing self-referentiality. Interestingly their features begin to exceed capabilities of the left brain: graphics dominates above scripture, associativity is implemented by google, social features are advocated by web2.0.

5. Summary and Future Work

Summary: TODO
Future work: TODO

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