Brigding Highschool — University in Mathematics
An Endless Saga

Walther Neuper

IICM Institute for Information Systems and Computer Media
Graz University of Technology

Minisymposium at AUSHUN15
“Proving in Math Education at University and at School”
Aug. 26, 2015 Gjör, Hungary
1. The gap between highschool $\leftrightarrow$ university . . .
   . . . is concern of EU policies
   . . . is being narrowed from both sides
   . . . is a matter for formality

   . . . by teaching more math in highschool?
   . . . by adapting university math?
   . . . by directly addressing formality?!!!

3. Discussion
   . . .
The big gap
EU policies
Narrowing initiatives
Crucial formality

1. The gap between highschool $\leftarrow\rightarrow$ university . . .
   . . . is concern of EU policies
   . . . is being narrowed from both sides
   . . . is a matter for formality

   . . . by teaching more math in highschool?
   . . . by adapting university math?
   . . . by directly addressing formality ?!!!

3. Discussion
   . . .
Highschool – University
Walther Neuper

The Rocard Report

The big gap
EU policies
Narrowing initiatives
Crucial formality

Bridge the gap
more math in highschool
Adapting university math
Addressing formality

Discussion

Science Education NOW: A Renewed Pedagogy for the Future of Europe

\[ f(x) = \cos(x^2) + 1 \]
Executive summary

In recent years, many studies have highlighted an alarming decline in young people’s interest for key science studies and mathematics. Despite the numerous projects and actions that are being implemented to reverse this trend, the signs of improvement are still modest. Unless more effective action is taken, Europe’s longer-term capacity to innovate, and the quality of its research will...

The findings and recommendations of the experts group are summarized below.

A reversal of school science-teaching pedagogy from mainly deductive to inquiry-based methods provides the means to increase interest in science.

Inquiry-based science education (IBSE) has proved its efficacy at both primary and secondary levels in increasing children’s and students’ interest and attainments levels while at the same time stimulating teacher motivation. IBSE is effective with all kinds of students from the weakest...
The gap between highschool $\Leftarrow \Rightarrow$ university ... 

... is concern of EU policies

... is being narrowed from both sides

... is a matter for formality

How bridge the gap? ... 

... by teaching more math in highschool?

... by adapting university math?

... by directly addressing formality ?!!!

Discussion
Attracts students from 8th semester, requires . . .

... much of mathematics taught in semesters 1, 2, while students "forget" the stuff meanwhile.
Attracts students from 8th semester, requires...

...much of mathematics taught in semesters 1, 2, while students "forget" the stuff meanwhile.
The big gap
EU policies
Narrowing initiatives
Crucial formality

Bridge the gap
more math in highschool
Adapting university math
Addressing formality

Discussion

1. The gap between highschool \( \rightarrow \) university . . .
   - . . . is concern of EU policies
   - . . . is being narrowed from both sides
   - . . . is a matter for formality

   - . . . by teaching more math in highschool?
   - . . . by adapting university math?
   - . . . by directly addressing formality ?!!!

3. Discussion
   . . .
Formality of math increases:

1. mathematics develops into increasing complexity  
   (also applications in engineering, . . .)
2. complexity is mastered by abstraction
3. abstraction is mastered by formality  
   (ring, complete field, skew field of quaterions, . . .)

How do students cope with formality?
Formality of math increases:

1. mathematics develops into increasing complexity
   (also applications in engineering, \ldots)

2. complexity is mastered by abstraction

3. abstraction is mastered by formality
   (ring, complete field, skew field of quaterions, \ldots)

How do students cope with formality?
Formality of math increases:

1. mathematics develops into increasing complexity (also applications in engineering, \ldots)
2. complexity is mastered by abstraction
3. abstraction is mastered by formality (ring, complete field, skew field of quaterions, \ldots)

How do students cope with formality?
Formality of math increases:

1. mathematics develops into increasing complexity
   (also applications in engineering, . . .)
2. complexity is mastered by abstraction
3. abstraction is mastered by formality
   (ring, complete field, skew field of quaterions, . . .)

How do students cope with formality?
The big gap
EU policies
Narrowing initiatives
Crucial formality

Bridge the gap
more math in highschool
Adapting university math
Addressing formality

Discussion

Questionnaire from 7 EU Univ.

The big gap
EU policies
Narrowing initiatives
Crucial formality

Bridge the gap
more math in highschool
Adapting university math
Addressing formality

Discussion

Questionnaire from 7 EU Univ.
Questionnaire from 7 EU Univ.

This page is about using laws to justify steps in simplifications. We give the following abbreviations for laws:

\[ [C+] \quad a + b = b + a \]
\[ [A++] \quad (a + b) + c = a + (b + c) \]
\[ [A+-] \quad (a + b) - c = a + (b - c) \]
\[ [U+] \quad a + 0 = a \]
\[ [D+] \quad a \cdot (b + c) = a \cdot b + a \cdot c \]
\[ [C\cdot] \quad a \cdot b = b \cdot a \]
\[ [A\cdot] \quad (a \cdot b) \cdot c = a \cdot (b \cdot c) \]
\[ [I+] \quad a - a = 0 \]
\[ [U\cdot] \quad a \cdot 1 = a \]
\[ [D-] \quad a \cdot (b - c) = a \cdot b - a \cdot c \]

Here is an example of stepwise justifying a simplification by use of these laws and by calculating natural numbers \([N+ - \cdot]\):

\[ 2 \cdot (x + 3 \cdot y) - 6 \cdot y = [D+] \quad (2 \cdot x + 2 \cdot (3 \cdot y)) - 6 \cdot y = [A\cdot] \quad (2 \cdot x + (2 \cdot 3) \cdot y) - 6 \cdot y = [N+] \quad (2 \cdot x + 6 \cdot y) - 6 \cdot y = [A+] \quad 2 \cdot x + (6 \cdot y - 6 \cdot y) = [I+] \quad 2 \cdot x + 0 = [U+] \quad 2 \cdot x. \]

4) **Similarly describe a stepwise justification of the following simplifications, please;**

Take as many steps you need:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = \]

\[ = \]

\[ = \]

\[ = \]

\[ = \]

b) \[ r + r \cdot (2 + s) = \]

\[ = \]

\[ = \]
Questionnaire from 7 EU Univ.

4) Similarly describe a stepwise justification of the following simplifications, please; Take as many steps you need:

a) \(2 \cdot (x + 3 \cdot y) + 6 \cdot y = \)
\[
\begin{align*}
&= (2 \cdot x + 2 \cdot (3 \cdot y)) + 6 \cdot y \\
&= 2 \cdot x + 6 \cdot y + 6 \cdot y \\
&= 2 \cdot x + (6 \cdot y + 6 \cdot y) \\
&= 2 \cdot x + 12 \cdot y
\end{align*}
\]

b) \(r + r \cdot (2 + s) = \)
\[
\begin{align*}
&= r + r \cdot 2 + r \cdot s \\
&= r \cdot (1 + 2) + r \cdot s \\
&= 3 \cdot r + s \cdot r \\
&= (3 + s) \cdot r
\end{align*}
\]

c) \((u + 1) \cdot (u - 1) = \)
\[
\begin{align*}
&= (u + 1) \cdot u - (u + 1) \cdot 1 \\
&= u \cdot (u + 1) - (u + 1) \\
&= u \cdot u + u - (u + 1) \\
&= u \cdot u + u - u - 1
\end{align*}
\]

5) Can the simplification \((x+y) \cdot (x-y) = x \cdot x - y \cdot y\) be justified using the above laws only?

a) If “yes”, give the first three steps and justifications, please:

\[(x+y)(x-y) = \]
\[
\begin{align*}
&= x \cdot (x+y) - y \cdot (x+y) \\
&= x \cdot x + x \cdot y - y \cdot (x+y) \\
&= x \cdot x + (x \cdot y - y \cdot x) + y \cdot y
\end{align*}
\]

b) If “no”, give some missing laws, please:
Questionnaire from 7 EU Univ.

4) Begründen Sie bitte ebenso beim schrittweise Vereinfachen; machen Sie soviele Schritte wie Sie brauchen:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = \]
\[ = \]
\[ = \]
\[ = \]
\[ = 2 \cdot x + 12 \cdot y \]

b) \[ r + r \cdot (2 + s) = \]
\[ = \]
\[ = \]
\[ = \]
\[ = 3r + 2s \]

c) \[ (u + 1) \cdot (u - 1) = \]
\[ = \]
\[ = \]
\[ = \]
\[ = u^2 - 1 \]

5) Lässt sich \((x+y)(x-y) = x \cdot x - y \cdot y\) nur mit obigen Gesetzen begründen?

a) Wenn “ja”, geben Sie bitte die ersten drei Schritte samt Begründung an:
\[ \]
\[ \]
\[ \]
\[ \]

b) Wenn “nein”, geben Sie bitte einige der fehlenden Gesetze an:
4) **Begründen Sie bitte ebenso beim schrittweisen Vereinfachen; machen Sie soviele Schritte wie Sie brauchen:**

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = [D^+] = 2x + 6y + 6y \]
   \[ = [A^{++}] = 2x + 12y \]
   \[ = [] = \]
   \[ = [ ... ] = \]

b) \[ r + r \cdot (2 + s) = [D^+] = y + 2r + r \cdot s \]
   \[ = [A^{++}] = 3y + r \cdot s \]
   \[ = [D^+] = \frac{y \cdot (s + s)}{[ ... ] =} \]

5) **Lässt sich (x+y) \cdot (x-y) = x \cdot x - y \cdot y nur mit obigen Gesetzen begründen?**

   a) Wenn „ja“, geben Sie bitte die ersten drei Schritte samt Begründung an:
   
   \[ (x+y)(x-y) = [D^-] = \kappa^2 - \kappa \cdot y + y \cdot (x-y) \]
   \[ = [A^+] = \kappa^2 - y^2 \]
   \[ = [ ... ] = \]

   b) Wenn „nein“, geben Sie bitte einige der fehlenden Gesetze an:
Questionnaire from 7 EU Univ.

4) Tomando la simplificación anterior como ejemplo, simplifica las siguientes expresiones algebraicas indicando en casa paso la regla usada. Usa tantos pasos como consideres necesarios:

a) \( 2\cdot(x + 3\cdot y) + 6\cdot y = \) \( [D+] = 2x + 6y + 6y \)
   \( = [.....] = 2x + 12y \)
   \( = [.....] = \) \( [.....] = \)

b) \( r + r(2 + s) = \) \( [D+] = r + 2r + rs \)
   \( = [D+] = 3r + rs \)
   \( = [D+] = (3+s)r \)
   \( = [.....] = \) \( [.....] = \)

c) \( (u + 1)\cdot(u - 1) = \) \( [.....] = u^2 - 1 \)
   \( = [.....] = \) \( [.....] = \) \( [.....] = \) \( [.....] = \)

5) ¿Puede la transformación \( (x + y) \cdot (x - y) = x\cdot x - y\cdot y \) justificarse usando únicamente las reglas mencionadas más arriba?

a) Si tu respuesta es “sí”, escribe los tres primeros pasos indicando la regla usada en cada paso:
   \( (x+y)(x-y) = \) \( [D+] = (x\cdot x - x\cdot y) + (y\cdot x - y\cdot y^2) \)
   \( = [.....] = x^2 - y^2 \) \( = [.....] = \) \( [.....] = \)
Math without formality?

Result from 350 questionnaires from 7 Universities from all over EU:
70% of STEM students cannot reliably apply rules.

Uncertainty in using formulas allows for
- for talking about math
- for standardised application of math
- not for inventive application of math
- not for creating new math knowledge.

Students do not like what they cannot do!
Math without formality?

Result from 350 questionnaires from 7 Universities from all over EU:
70% of STEM students cannot reliably apply rules.

Uncertainty in using formulas allows for

- for talking about math
- for standardised application of math
  - not for inventive application of math
  - not for creating new math knowledge.

Students do not like what they cannot do!
Result from 350 questionnaires from 7 Universities from all over EU: 70% of STEM students cannot reliably apply rules.

Uncertainty in using formulas allows for

- for talking about math
- for standardised application of math
- not for inventive application of math
- not for creating new math knowledge.

Students do not like what they cannot do!
Math without formality?

Result from 350 questionnaires from 7 Universities from all over EU:
70% of STEM students cannot reliably apply rules.

Uncertainty in using formulas allows for
- for talking about math
- for standardised application of math
- not for inventive application of math
- not for creating new math knowledge.

Students do not like what they cannot do!
Isn’t here the essence of the gap?

<table>
<thead>
<tr>
<th></th>
<th>natural language</th>
<th>formal language</th>
</tr>
</thead>
<tbody>
<tr>
<td>syntax</td>
<td>by rules</td>
<td>by formal rules</td>
</tr>
<tr>
<td>grammar</td>
<td>by rules</td>
<td>by formal rules</td>
</tr>
<tr>
<td>semantics</td>
<td>by human interaction</td>
<td>by formulas</td>
</tr>
<tr>
<td>argumentation</td>
<td>by human interaction</td>
<td>by formal proof</td>
</tr>
</tbody>
</table>
The gap between highschool $\leftrightarrow$ university ... 
... is concern of EU policies
... is being narrowed from both sides
... is a matter for formality

How bridge the gap? ...
... by teaching more math in highschool?
... by adapting university math?
... by directly addressing formality?!!!

Discussion
...by teaching more proof?

These are the prerequisites to understand proofs:

- variable (functions with bound variables)
- apply rules formally
- axiom — definition — theorem
- proof

...all in a reasonable variety of experience.

Realistically: How far can we get at highschool?
... by teaching more proof?

These are the prerequisites to understand proofs:

- variable (functions with *bound* variables)
- apply rules formally
- axiom — definition — theorem
- proof

... all in a reasonable variety of experience.

Realistically: How far can we get at highschool?
...by teaching more proof?

These are the prerequisites to understand proofs:

- variable (functions with *bound* variables)
- apply rules formally
- axiom — definition — theorem
- proof

...all in a reasonable variety of experience.

Realistically: How far can we get at highschool?
by teaching more proof?

These are the prerequisites to understand proofs:

- variable (functions with \textit{bound} variables)
- apply rules formally
- axiom — definition — theorem
- proof

... all in a reasonable variety of experience.

Realistically: How far can we get at highschool?
...by teaching more proof?

These are the prerequisites to understand proofs:

- variable (functions with \textit{bound} variables)
- apply rules formally
- axiom — definition — theorem
- proof

...all in a reasonable variety of experience.

Realistically: How far can we get at highschool?
...by teaching more proof?

These are the prerequisites to understand proofs:
• variable (functions with bound variables)
• apply rules formally
• axiom — definition — theorem
• proof

...all in a reasonable variety of experience.

Realistically: How far can we get at highschool?
The big gap
EU policies
Narrowing initiatives
Crucial formality

Bridge the gap
more math in highschool
Adapting university math
Addressing formality

Discussion

1. The gap between highschool $\leftarrow\rightarrow$ university ...
   ... is concern of EU policies
   ... is being narrowed from both sides
   ... is a matter for formality

2. How bridge the gap? ...
   ... by teaching more math in highschool?
   ... by adapting university math?
   ... by directly addressing formality?!!!

3. Discussion
   ...
Lots of "Introduction to Mathematics"

infinite number of lecture notes
The gap between highschool $\leftarrow\rightarrow$ university ... 
... is concern of EU policies
... is being narrowed from both sides
... is a matter for formality

How bridge the gap ? ... 
... by teaching more math in highschool?
... by adapting university math?
... by directly addressing formality ?!!!

Discussion
...
Directly addressing formality

Given the example . . .

\[ \cdots = \frac{d}{dx} x + \frac{d}{dx} \sin(x^2) = \frac{d}{dx} \sin(\theta a) = \cos(\theta a) \times \frac{d}{dx} \theta a \]

\[ = \frac{d}{dx} x + \cos(x^2) \times \frac{d}{dx} (x^2) = \cdots \]

... look at the formal structure ...
Directly addressing formality

...this can be addressed dynamically this way ...

\[ \cdots = \frac{d}{dx} x + \frac{d}{dx} \sin(x^2) = \]
\[ \frac{d}{dx} \sin(x^2) = \cos(x^2) \frac{d}{dx} x \]
\[ = [] \]

...of that way ...

\[ \cdots = \frac{d}{dx} x + \frac{d}{dx} \sin(x^2) = \]
\[ \frac{d}{dx} \sin(x^2) = \cos(x^2) \frac{d}{dx} x \]
\[ = \frac{d}{dx} x + \cos(x^2) \frac{d}{dx} [] \cdots \]

...or . . . [] is the cursor.
Directly addressing formality

This way addressing formality by specific software,

which models *stepwise problem solving*

similar to what is written to blackboard or exercise book

and which addresses such formalities during problem solving *incidentally and concurrently* (and *not* as a separate exercise) —

— what is the impact of such specific software?
How does it integrate with other initiatives?
Directly addressing formality

This way addressing formality by specific software, which models \textit{stepwise problem solving} similar to what is written to blackboard or exercise book and which addresses such formalities during problem solving \textit{incidentally and concurrently} (and \textit{not} as a separate exercise) —

— what is the impact of such specific software? How does it integrate with other initiatives?
Directly addressing formality

This way addressing formality by specific software, which models *stepwise problem solving* similar to what is written to blackboard or exercise book and which addresses such formalities during problem solving *incidentally and concurrently* (and not as a separate exercise) —

— what is the impact of such specific software? How does it integrate with other initiatives?
Directly addressing formality

This way addressing formality by specific software,

which models *stepwise problem solving*

similar to what is written to blackboard or exercise book

and which addresses such formalities during problem solving *incidentally and concurrently* (and *not* as a separate exercise) —

— what is the impact of such specific software?

How does it integrate with other initiatives?
And does such specific software meet the challenges of formality?…

<table>
<thead>
<tr>
<th>natural language</th>
<th>formal language</th>
</tr>
</thead>
<tbody>
<tr>
<td>syntax</td>
<td>by rules</td>
</tr>
<tr>
<td>grammar</td>
<td>by rules</td>
</tr>
<tr>
<td>semantics</td>
<td>by human interaction</td>
</tr>
<tr>
<td>argumentation</td>
<td>by human interaction</td>
</tr>
</tbody>
</table>
1. The gap between highschool $\leftarrow \rightarrow$ university ...  
   ... is concern of EU policies  
   ... is being narrowed from both sides  
   ... is a matter for formality

2. How bridge the gap? ...  
   ... by teaching more math in highschool?  
   ... by adapting university math?  
   ... by directly addressing formality?!!!

3. Discussion  
   ...
Thank you for attention!

Opinions?! Questions?!