Promises of Computer-Theorem-Prover technology for education

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Proving in Mathematics Education at University and at School
Minisymposion at CSASC
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Outline
EU-wide Questionnaire

Theorems justifying steps in algebraic transformations

University of .................................. (if possible, change icon accordingly)

Mathematics is considered a difficult subject. This questionnaire is part of a research on basic principles and reasons for difficulties in mathematics education. Thus you are just asked basic knowledge, most of which will remind you of your early mathematics classes.

1) Below you find basic laws of algebra (i.e. theorems). (a) Do you remember some laws? (b) Do you even remember the names of some laws? (c) Can you apply the laws to numbers?

   a) \( a + b = b + a \) (a) \( \n \) (b) law of commutativity for +
   b) \( a \cdot b = b \cdot a \) (\( \n \) no)
   c) \( (a + b) + c = a + (b + c) \) (\( \n \) yes) no law
   d) \( a \cdot (b + c) = a \cdot b + a \cdot c \) (\( \n \) yes) no law
   e) \( a + 0 = a \) (\( \n \) yes)
   f) \( a + 0 = a \) (\( \n \) yes)

2) Do you remember any other mathematical laws, not yet mentioned above?

   a) .............................................. law
   b) .............................................. law
   c) .............................................. law

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra, but usually one simplifies without laws, for instance:

   a) \( 2(x + 3y) - 6y - 2x + 6y - 6y = \) ...
   b) \( 2(x + 3y) + 6y = \) ...
   c) \( r + r(2 + x) = \) ...
   d) \( (u + 1) \cdot (u - 1) = \) ...
   e) \( (x + y) \cdot (x - y) = \) ...

4) Similarly describe a stepwise justification of the following simplifications, please; take as many steps you need:

   a) \( 2(x + 3y) + 6y \) ...
   b) \( r + r(2 + x) \) ...
   c) \( (u + 1)(u - 1) \) ...

5) Can the simplification \( (x+y)(x-y) = x^2 - y^2 \) be justified using the above laws only?

   a) If "yes", give the first three steps and justifications, please:
       \( (x+y)(x-y) = x^2 - y^2 \) ...
   b) If "no", give some missing laws, please:

6) I am a student in a degree course for mathematics (at university) \( \n \) yes \( \n \) no

Thank you for your attention!

For results see http://www.ist.tugraz.at/projects/issac/www/content/status.html#quest10
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a) \[ a + b = b + a \] (X yes ☐ no) (b) law of commutativity for +
(b) \[ a \cdot b = b \cdot a \] ☐ yes ☐ no  
(c) \[ (a + b) + c = a + (b + c) \] ☐ yes ☐ no 
(d) \[ a \cdot (b + c) = a \cdot b + a \cdot c \] ☐ yes ☐ no 
(e) \[ a + 0 = a \] ☐ yes ☐ no 
(f) \[ a - 0 = a \] ☐ yes ☐ no 

2) Do you remember any other mathematical laws, not yet mentioned above?

a) .................................................. law ..................................................
(b) .................................................. law ..................................................
(c) ..................................................

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra, but usually one simplifies without laws, for instance:

a) \[ 2(x + 3y) - 6y = 2x + 6y - 6y = 2x \]
b) \[ 2(x + 3y) + 6y = \]
c) \[ r + r(2 + s) = \]
d) \[ (u + 1) \cdot (u - 1) = \]
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5) Can the simplification \((x+y)(x-y)=x^2-y^2\) be justified using the above laws only?

a) If "yes", give the first three steps and justifications, please:

\[
(x+y)(x-y) = x^2 - y^2 = 
\]

b) If "no", give some missing laws, please:

..................................................

..................................................

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6) I am a student in a degree course for mathematics at university ☐ yes ☐ no

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   a) \(a + b = b + a\) (X yes □ no) (b) law of commutativity for +
   (c) 2 + 3 = 3 + 2 ... 5 = 5
   b) \(a \cdot b = b \cdot a\) □ yes □ no law __________________________
   c) \((a + b) + c = a + (b + c)\) □ yes □ no law __________________________
   d) \(a \cdot (b + c) = a \cdot b + a \cdot c\) □ yes □ no law __________________________
   e) \(a + 0 = a\) □ yes □ no law __________________________
   f) \(a - a = 0\) □ yes □ no law __________________________

2) Do you remember any other mathematical laws, not yet mentioned above?

   a) ___________________________ law __________________________
   b) ___________________________ law __________________________
   c) ___________________________ law __________________________

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra; but usually one simplifies without laws, for instance:

   a) \(2(x + 3y) - 6y - 2x + 6y - 6y = \ldots\)
   b) \(2(x + 3y) + 6y = \ldots\)
   c) \(r + r(2 + s) = \ldots\)
   d) \((a + 1) \cdot (u - 1) = \ldots\)
   e) \((x + y) \cdot (x - y) = \ldots\)

4) Similarly describe a stepwise justification of the following simplifications, please; Take as many steps as you need:

   a) \(2(x + 3y) + 6y = \ldots\)
   b) \(r + r(2 + s) = \ldots\)
   c) \((u + 1) \cdot (u - 1) = \ldots\)

5) Can the simplification \((x + y)(x - y) = x^2 - y^2\) be justified using the above laws only?

   a) If "yes", give the first three steps and justifications, please:
      \((x + y)(x - y) = \ldots\)
   b) If "no", give some missing laws, please:
      ___________________________ law __________________________
      ___________________________ law __________________________
      ___________________________ law __________________________

6) I am a student in a degree course for mathematics (at university) □ yes □ no

Thank you for your attention!

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Students can simplify

\[ \ldots = \ldots \quad \ldots = \ldots \]

\[ g) \quad a - a = 0 \quad \square \text{yes} \quad \square \text{no} \quad \text{law} \]

\[ \ldots = \ldots \quad \ldots = \ldots \]

2) Do you remember any other mathematical laws, not yet mentioned above?

a) \[ \text{law} \]

b) \[ \text{law} \]

c) \[ \text{law} \]

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra; but usually one simplifies without laws, for instance:

a) \[ 2 \cdot (x + 3 \cdot y) - 6y = 2 \cdot x + 6 \cdot y - 6 \cdot y = 2 \cdot x \]

b) \[ 2 \cdot (x + 3 \cdot y) + 6y = \text{expression} = \]

c) \[ r + r \cdot (2 + s) = \text{expression} = \]

d) \[ (u + 1) \cdot (u - 1) = \text{expression} = \]

e) \[ (x + y) \cdot (x - y) = \text{expression} = \]
Students **can simplify**

2) **Erinnern Sie sich auch an andere Gesetze (die Sie nicht schon oberhalb finden)?**
   a) 
   
   b) 
   
   c) 

3) **Vereinfachen Sie bitte die folgenden Ausdrücke!** Rechengesetze werden zusammen mit solchen Vereinfachungen gelernt; diese gelingen auch ohne Rechengesetze richtig, z.B.:
   a) \(2 \cdot (x + 3 \cdot y) - 6y = 2 \cdot x + 6 \cdot y - 6 \cdot y = 2 \cdot x\)
   b) \(2 \cdot (x + 3 \cdot y) + 6y = 2x + 6y + 6y = 2x + 12y\)
   c) \(r + r \cdot (2 + s) = r + 2r + rs\)
   d) \((u + 1) \cdot (u - 1) = u^2 + u - u - 1\)
   e) \((x + y) \cdot (x - y) = x^2 + xy - xy - y^2\)
2) ¿Te acuerdas de alguna regla matemática no mencionada en el apartado anterior?
   a) ___________  
      elemente inverso  
      regla ___________  
   b) ___________  
      ___________  
      regla ___________  
   c) ___________  
      ___________  
      regla ___________  

3) Simplifica las siguientes expresiones algebraicas. La simplificación de expresiones se aprende habitualmente junto con las reglas algebraicas, pero por lo general tendemos a simplificar sin pensar en las reglas, por ejemplo:
   a) 2 · (x + 3 · y) − 6y = 2 · x + 6 · y − 6 · y = \(2x\)  
   b) 2 · (x + 3 · y) + 6y = \(2x + 6y + 6y\)  
   c) \(r + r(2 + s)\) = \(3r + rs\)  
   d) \((u + 1) · (u - 1)\) = \(u^2 - u + u - 1\)  
   e) \((x + y) · (x - y)\) = \(x^2 - xy + xy - y^2\)
Students can simplify

1) \( a - a = 0 \)  \( \text{(da)} \)  ne  \( 5 - 5 = 0 \)  \( \text{zakon} \)  

2) Možete li navesti još neke zakone koji nisu gore navedni?
   a) \______________
   b) \______________
   c) \______________
   \( \text{zakon} \)  

3) Uprostite naredne izraze.:
   Primer: \( 2 \cdot (x + 3 \cdot y) - 6y = 2 \cdot x + 6 \cdot y - 6 \cdot y = 2 \cdot x \)
   a) \( 2 \cdot (x + 3 \cdot y) + 6y = \frac{2 \cdot x + 6 \cdot y + 6 \cdot y}{\text{zakon}} = 2x + 12y \)
   b) \( r + r \cdot (2 + s) = \frac{r + 2r + 2s}{\text{zakon}} = 3r + 2s \)
   c) \( (u + 1) \cdot (u - 1) = \frac{u^2 - 1}{\text{zakon}} = u^2 - 1 \)
   d) \( (x + y) \cdot (x - y) = \frac{x^2 - y^2}{\text{zakon}} = x^2 - y^2 \)
Students can simplify

2) Which laws (theorems) not yet mentioned above do you remember?
   a) ................................................................. law .........................................................
   b) ................................................................. law .........................................................
   c) ................................................................. law .........................................................

3) Algebraic laws are learned together with transformations generally done without using laws, for instance:
   a) \(2 \cdot (x + 3 \cdot y) - 6y = 2 \cdot x + 6 \cdot y - 6 \cdot y = 2 \cdot x\)
   b) \(2 \cdot (x + 3 \cdot y) + 6y = \frac{2 \cdot x + 6 \cdot y - 6 \cdot y}{(1 \cdot 2 \cdot x + 6 \cdot y - 6 \cdot y)} = \frac{2 \cdot x + 6 \cdot y}{3 \cdot y + 5}\)
   c) \(r + r \cdot (2 + s) = \frac{x + 2 \cdot r + s}{(1 \cdot r + s)} = \frac{3 \cdot r + s}{s}\)
   d) \((u + 1) \cdot (u - 1) = \frac{u + (u + 1) - (u + 1) - u}{u + (u + 1) - (u + 1) - u} = \frac{u^2 - 1}{u}\)
   e) \((x + y) \cdot (x - y) = \frac{x \cdot (x - y) + y \cdot (x - y)}{x \cdot (x - y) + y \cdot (x - y)} = \frac{x^2 - y^2}{x^2 - y^2}\)
Students can simplify

- Front-side: students can simplify
  - They perfectly operate on formulas
  - Do they have a notion of 'theorem'?
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Students know theorems
Students know theorems

Knowledge, most of which will remind you of early mathematics classes.

1) Below you find basic laws of algebra (i.e. theorems). (a) **Do you remember some laws?** (b) Do you even remember the names of some laws? (c) Can you apply the laws to numbers?

   a) \( a + b = b + a \)  
      
      (a) \( \times \) yes \( \Box \) no  
      (b) law of commutativity for +
      
      (c) \( 2 + 3 = 3 + 2 \) \( \ldots \) \( 5 = 5 \)

   b) \( a \cdot b = b \cdot a \)

      \( \Box \) yes \( \Box \) no  
      law ........................................
      
      \( \ldots \cdot = \ldots \cdot \ldots \ldots \ldots \ldots = \ldots \)

   c) \( (a + b) + c = a + (b + c) \)

      \( \Box \) yes \( \Box \) no  
      law ........................................
      
      \( \ldots \ldots \ldots = \ldots \ldots \ldots \ldots = \ldots \ldots \ldots \ldots = \ldots \ldots \ldots \ldots = \ldots \)

   d) \( a \cdot (b + c) = a \cdot b + a \cdot c \)

      \( \Box \) yes \( \Box \) no  
      law ........................................
      
      \( \ldots \ldots \ldots = \ldots \ldots \ldots \ldots = \ldots \ldots \ldots \ldots = \ldots \ldots \ldots \ldots = \ldots \)

   e) \( a \cdot 1 = a \)

      \( \Box \) yes \( \Box \) no  
      law ........................................
      
      \( \ldots \ldots = \ldots \ldots \ldots = \ldots \ldots \ldots = \ldots \ldots \ldots = \ldots \ldots \ldots = \ldots \)

   f) \( a + 0 = a \)

      \( \Box \) yes \( \Box \) no  
      law ........................................
      
      \( \ldots \ldots = \ldots \ldots \ldots = \ldots \ldots \ldots = \ldots \ldots \ldots = \ldots \ldots \ldots = \ldots \)

   g) \( a - a = 0 \)

      \( \Box \) yes \( \Box \) no  
      law ........................................
      
      \( \ldots \ldots = \ldots \ldots \ldots = \ldots \ldots \ldots = \ldots \ldots \ldots = \ldots \ldots \ldots = \ldots \ldots \ldots = \ldots \)
Students know theorems

Knowledge, most of which will remind you of early mathematics classes.

1) Below you find basic laws of arithmetic (i.e. theorems). Do you know any of these laws? Do you even remember the names of some laws? Can you apply the laws to numbers?

   a + b = b + a  \[ \checkmark \text{yes} \square \text{no} \]  law of commutativity for +
   \[ 2 + 3 = 3 + 2 \ldots 5 = 5 \]

   b) \ a \cdot b = b \cdot a \quad \[ \checkmark \text{yes} \square \text{no} \]  law \underline{Kommutativ} \text{?}
   \[ 2 \cdot 3 = 3 \cdot 2 \ldots 6 = 6 \]

   c) \ (a + b) + c = a + (b + c) \quad \[ \checkmark \text{yes} \square \text{no} \]  law \underline{Assoziativ} \text{?}
   \[ (1+2)+3 = 1+(2+3) \ldots 3+3 = 1+5 \ldots 6 = 6 \]

   d) \ a \cdot (b + c) = a \cdot b + a \cdot c \quad \[ \checkmark \text{yes} \square \text{no} \]  law \underline{Distributiv} \text{?}
   \[ 1 \cdot (2+3) = 1 \cdot 2 + 1 \cdot 3 \ldots 5 = 2+3 \ldots 5 = 5 \]

   e) \ a \cdot 1 = a \quad \[ \checkmark \text{yes} \square \text{no} \]  law \underline{Neutrals} \text{?}
   \[ 5 \cdot 1 = 5 \ldots 5 = 5 \]

   f) \ a + 0 = a \quad \[ \checkmark \text{yes} \square \text{no} \]  law \underline{Neutrals} \text{?}
   \[ 3+0 = 3 \ldots 3 = 3 \]

   g) \ a - a = 0 \quad \[ \checkmark \text{yes} \square \text{no} \]  law \underline{Neutrals} \text{?}
   \[ 3-3 = 0 \ldots 0 = 0 \]

2) Which laws (theorems) not yet mentioned above do you remember?
1) Hier sind einfache Rechengesetze. (i) Können Sie sich an einige Rechengesetze erinnern? (ii) Wissen Sie ihre Namen noch? (iii) Können Sie die Gesetze auf Zahlen anwenden?

 a) \( a + b = b + a \)  
   (i) \( X \) ja □ nein  
   (ii) Gesetz der Vertauschung für + 
   (iii) \( 2 + 3 = 3 + 2 \) ... \( 5 = 5 \)

 b) \( a \cdot b = b \cdot a \)  
   □ ja □ nein  
   Gesetz der Vertauschung für .
   \( 2 \cdot 3 = 3 \cdot 2 \) ... \( 6 = 6 \).

c) \( (a + b) + c = a + (b + c) \)  
   □ ja □ nein  
   Gesetz ________________________.
   \( (1 + 2) + 3 = 1 + (2 + 3) \) ... \( 6 = 6 \) ... ... = ...

d) \( a \cdot (b + c) = a \cdot b + a \cdot c \)  
   □ ja □ nein  
   Gesetz ________________________.
   \( 2 \cdot (1 + 3) = 2 \cdot 1 + 2 \cdot 3 \) ... \( 8 = 8 \).

e) \( a \cdot 1 = a \)  
   □ ja □ nein  
   Gesetz ________________________.
   \( 4 \cdot 1 = 4 \) ... \( 4 = 4 \).

 f) \( a + 0 = a \)  
   □ ja □ nein  
   Gesetz ________________________.
   \( 2 + 0 = 2 \) ... \( 2 = 2 \).

g) \( a - a = 0 \)  
   □ ja □ nein  
   Gesetz ________________________.
   \( 1 - 1 = 0 \) ... \( 0 = 0 \).

2) Erinnern Sie sich auch an andere Gesetze (die Sie nicht schon oben genannt haben)?
1) A continuación encontrarás las reglas básicas del álgebra (es decir, teoremas).
(a) ¿Te acuerdas de alguna regla? (b) ¿Te acuerdas del nombre de alguna regla? (c) ¿Sabrías aplicar las reglas en casos concretos?

(a) \[ X \] sí \[ \square \] no (b) regla conmutatividad de la suma
\[ a + b = b + a \]
(c) 2 + 3 = 3 + 2 \[ \ldots \] 5 = 5

(b) \[ \square \] sí \[ \square \] no regla conmutatividad del producto
\[ a \cdot b = b \cdot a \]
3 \[ \cdot \] 2 = 2 \[ \cdot \] 3 \[ \ldots \] 6 = 6.

(c) \[ \square \] sí \[ \square \] no regla propiedad asociativa
\[ (a + b) + c = a + (b + c) \]
\[ (3 + 2) + 5 = 3 + (2 + 5) \] \[ \ldots \] 5 + 5 = 3 + 7 \[ \ldots \] 10 = 10

(d) \[ \square \] sí \[ \square \] no regla propiedad distributiva
\[ 2 \cdot (3 + 4) = 2 \cdot 3 + 2 \cdot 4 \] \[ \ldots \] 2 \cdot 7 = 6 + 8 \[ \ldots \] 14 = 14

(e) \[ \square \] sí \[ \square \] no regla elemento neutro del producto
\[ 5 \cdot 1 = 5 \] \[ \ldots \] 5 = 5.

(f) \[ \square \] sí \[ \square \] no regla elemento neutro de la suma
\[ 4 + 0 = 4 \] \[ \ldots \] 4 = 4.

(g) \[ \square \] sí \[ \square \] no regla elemento opuesto
\[ 1 - 1 = 0 \] \[ \ldots \] 0 = 0.
Students know theorems

matematičkog obrazovanja.

1) U nasvaku su navedeni neki od osnovnih algebarskih zakona (teorema). (a) Da li se sećate nekih od ovih zakona? (b) Da li se sećate njihovih imena? (c) Umjeti li da primenite ove zakone na brojeve?

Primer: \( a + b = b + a \)

(a) da ne

(c) \( 2 + 3 = 3 + 2 \)

\[ 2 + 3 = 5 \]

(b) komutativnost operacije +

(a) da ne

(b) \( 2 + 3 = 3 + 2 \)

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\[ 2 + 3 = 5 \]

(b) komutativnost operacije +

(a) da ne

(b) \( 2 + 3 = 3 + 2 \)

\[ 2 + 3 = 5 \]

2) Možete li navesti još neke zakone koji nisu gore navedni?
Students know theorems

- Front-side: students can simplify
  - They perfectly operate on formulas
  - Do they have a notion of 'theorem'? 

- Front-side: students know theorems
  - They demonstrate comprehension of “theorem/rule”: one rule for all numbers
  - All great !!!!?
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Theorems justifying steps in algebraic transformations
University of .................................. (if possible, change icon accordingly)

Mathematics is considered a difficult subject. This questionnaire is part of a research on basic principles and reasons for difficulties in mathematics education. Thus you are just asked basic knowledge, most of which will remind you of early mathematics classes.

1) Below you find basic laws of algebra (i.e. theorems).
   (a) Do you remember some laws?
   (b) Do you even remember the names of some laws?
   (c) Can you apply the laws to numbers?

   a) \( a + b = b + a \)  
   \( \text{X yes} \)  no  
   (law of commutativity for +)
   b) \( a \cdot b = b \cdot a \)  
   \( \text{X yes} \)  no  
   (law)
   c) \( (a + b) + c = a + (b + c) \)  
   \( \text{X yes} \)  no  
   (law)
   d) \( a \cdot (b + c) = a \cdot b + a \cdot c \)  
   \( \text{X yes} \)  no  
   (law)
   e) \( a + 0 = a \)  
   \( \text{X yes} \)  no  
   (law)
   f) \( a - a = 0 \)  
   \( \text{X yes} \)  no  
   (law)

2) Do you remember any other mathematical laws, not yet mentioned above?
   a) ...........................................
   law
   b) ...........................................
   law
   c) ...........................................
   law

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra, but usually one simplifies without laws, for instance:
   a) \( 2(x + 3y) - 6y = 2x + 6y - 6y = 2x \)
   b) \( 2(x + 3y) + 6y = \)
   c) \( r + r(2 + s) = \)
   d) \( (u + 1) \cdot (u - 1) = \)
   e) \( (x + y) \cdot (x - y) = \)

4) Similarly describe a stepwise justification of the following simplifications, please;
   Take as many steps you need:
   a) \( 2(x + 3y) + 6y = \)
   b) \( r + r(2 + s) = \)
   c) \( (u + 1) \cdot (u - 1) = \)
   d) \( (x + y) \cdot (x - y) = \)

5) Can the simplification \( (x+y)(x-y) = x^2 - y^2 \) be justified using the above laws only?
   a) If "yes", give the first three steps and justifications, please:
   \( (x+y)(x-y) = \)
   \( x^2 - y^2 = \)
   b) If "no", give some missing laws, please:
   ...........................................
   law
   ...........................................
   law
   ...........................................
   law

6) I am a student in a degree course for mathematics (at university)  
   \( \text{X yes} \)  no

Thank you for your attention!

For results see http://www.ist.tugraz.at/projects/issac/www/content/status.html#quest10
This page is about using laws to justify steps in simplifications. We give the following abbreviations for laws:

- \([C+]\) \(a + b = b + a\)
- \([A++)\) \((a + b) + c = a + (b + c)\)
- \([A+-]\) \((a + b) - c = a + (b - c)\)
- \([U+]\) \(a + 0 = a\)
- \([D+]\) \(a \cdot (b + c) = a \cdot b + a \cdot c\)
- \([C-]\) \(a \cdot b = b \cdot a\)
- \([A-]\) \((a \cdot b) \cdot c = a \cdot (b \cdot c)\)
- \([I+]\) \(a - a = 0\)
- \([U-]\) \(a \cdot 1 = a\)
- \([D-]\) \(a \cdot (b - c) = a \cdot b - a \cdot c\)

Here is an example of stepwise justifying a simplification by use of these laws and by calculating natural numbers \([N+−\cdot]\):

\[
2 \cdot (x + 3 \cdot y) - 6 \cdot y \quad \overset{[D+]}{=} \quad (2 \cdot x + 2 \cdot (3 \cdot y)) - 6 \cdot y \quad \overset{[A-]}{=} \quad (2 \cdot x + (2 \cdot 3) \cdot y) - 6 \cdot y
\]

\[
= \overset{[N]}{=} \quad (2 \cdot x + 6 \cdot y) - 6 \cdot y \quad \overset{[A+−]}{=} \quad 2 \cdot x + (6 \cdot y - 6 \cdot y) \quad \overset{[I+]}{=} \quad 2 \cdot x + 0 \quad = [U+] = \quad 2 \cdot x
\]

4) **Similarly describe a stepwise justification of the following simplifications, please;**

   Take as many steps you need:

   a) \(2 \cdot (x + 3 \cdot y) + 6 \cdot y\) \(\overset{[\ldots\ldots]}{=}\) .................................................................

   \(\overset{[\ldots\ldots]}{=}\) .................................................................

   \(\overset{[\ldots\ldots]}{=}\) .................................................................

   \(\overset{[\ldots\ldots]}{=}\) .................................................................

   b) \(r + r \cdot (2 + s)\) \(\overset{[\ldots\ldots]}{=}\) .................................................................

   \(\overset{[\ldots\ldots]}{=}\) .................................................................
Theorems justifying steps in algebraic transformations

University of …………………. (if possible, change icon accordingly)

Mathematics is considered a difficult subject. This questionnaire is part of a research on basic principles and reasons for difficulties in mathematics education. Thus you are just asked basic knowledge, most of which will remind you of early mathematics classes.

1) Below you find basic laws of algebra (i.e. theorems). Do you recall any laws? (b) Do you even recognize the names of some laws? (c) Can you apply the laws to numbers?
   a) \( a + b = b + a \) \( \quad \) yes \( \checkmark \) no \( \quad \) law \( \text{commutativity of +} \)
   b) \( a \cdot b = b \cdot a \) \( \quad \) yes \( \checkmark \) no \( \quad \) law \( \text{commutativity of \cdot} \)
   c) \( (a + b) + c = a + (b + c) \) \( \quad \) yes \( \checkmark \) no \( \quad \) law \( \text{associativity of +} \)
   d) \( a(b + c) = ab + ac \) \( \quad \) yes \( \checkmark \) no \( \quad \) law \( \text{distributivity} \)
   e) \( a \cdot (b + c) = ab + ac \) \( \quad \) yes \( \checkmark \) no \( \quad \) law \( \text{distributivity} \)

2) Do you remember any other mathematical laws, not yet mentioned above?
   a) ............................................ law ...
   b) ............................................ law ...
   c) ............................................ law ...

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra, but usually one simplifies without laws, for instance:
   a) \( 2(x + 3y) - 6y = 2x + 6y - 6y = 2x \)
   b) \( 2(x + 3y) + 6y = 2x + 6y + 6y = 2x + 12y \)
   c) \( r + r(2 + 3) = r + r(5) = 2r \)
   d) \( (a + 1) \cdot (a - 1) = a^2 - 1 \)
   e) \( (x + y) \cdot (x - y) = x^2 - y^2 \)

4) Similarly describe a stepwise justification of the following simplifications, please; Take as many steps as needed:
   a) \( 2(x + 3y) + 6y \)
   b) \( r + r(2 + 3) \)
   c) \( (a + 1)(a - 1) \)

5) Can the simplification \( (x + y)(x - y) = x^2 - y^2 \) be justified using the above laws only? (a) If "yes", give the first three steps and justifications, please:
   \( (x+y)(x-y) = \)

(b) If "no", give some missing laws, please:

6) I am a student in a degree course for mathematics (at university) \( \checkmark \) yes \( \checkmark \) no

Thank you for your attention!

For results see http://www.ist.tugraz.at/projects/iasc/www/content/status.html/quest10
Cannot use theorems

5) Can the simplification \((x+y) \cdot (x-y) = x \cdot x - y \cdot y\) be justified using the above laws only?
   a) If “yes”, give the first three steps and justifications, please:
\[
(x+y)(x-y) = \boxed{\ldots} = \boxed{\ldots} = \boxed{\ldots}
\]
   b) If “no”, give some missing laws, please:

------------------------------- law -----------------------------
------------------------------- law -----------------------------
------------------------------- law -----------------------------

6) I am a student in a degree course for mathematics (at university) □ yes □ no

Thank you for your attention!
Cannot use theorems

4) Similarly describe a stepwise justification of the following simplifications, please; Take as many steps you need:

a) \(2 \cdot (x + 3 \cdot y) + 6 \cdot y\)

\[
\begin{align*}
2 \cdot (x + 3 \cdot y) + 6 \cdot y & \implies (2 \cdot x + 2 \cdot (3 \cdot y)) + 6 \cdot y \\
& \implies (2 \cdot x + (2 \cdot 3 \cdot y)) + 6 \cdot y \\
& \implies (2 \cdot x + 6 \cdot y) + 6 \cdot y = 2 \cdot x + (6 \cdot y + 6 \cdot y) \\
& \implies 2 \cdot x + 12 \cdot y
\end{align*}
\]

b) \(r + r \cdot (2 + s)\)

\[
\begin{align*}
r + r \cdot (2 + s) & \implies r + r \cdot 2 + r \cdot s \\
& \implies r \cdot (1 + 2) + r \cdot s \\
& \implies r \cdot 3 + r \cdot s \\
& \implies 3 \cdot r + s \cdot r
\end{align*}
\]

5) Can the simplification \((x+y) \cdot (x-y) = x \cdot x - y \cdot y\) be justified using the above laws only?

a) If “yes”, give the first three steps and justifications, please:

\[
\begin{align*}
(x+y) \cdot (x-y) & \implies x \cdot (x+y) - y \cdot (x+y) \\
& \implies x \cdot x + x \cdot y - y \cdot (x+y) \\
& \implies x \cdot x + (x \cdot y - y \cdot x+y)
\end{align*}
\]
**Cannot use theorems**

\[
2 \cdot (x + 3 \cdot y) + 6 \cdot y = [N] = (2 \cdot x + 6 \cdot y) - 6 \cdot y = [\text{A}^+] = 2 \cdot x + (6 \cdot y - 6 \cdot y) = [\text{I}^+] = 2 \cdot x + 0 = [\text{U}^+] = 2 \cdot x.
\]

4) **Begründen Sie bitte ebenso beim schrittweisen Vereinfachen; machen Sie soviele Schritte wie Sie brauchen:**

a) \[
2 \cdot (x + 3 \cdot y) + 6 \cdot y = [\text{D}^+] = \frac{2 \cdot x + 2 \cdot 3 \cdot y + 6 \cdot y}{2 \cdot x + 3 \cdot y + 6 \cdot y} = \frac{2 \cdot x + 12 \cdot y}{2 \cdot x + 12 \cdot y} = [\text{C}^+] = 2 \cdot x + 12 \cdot y.
\]

b) \[
r + r \cdot (2 + s) = [\text{D}^+] = r + (2 \cdot r + r \cdot s) = [\text{D}^+] = 3 \cdot r + r \cdot s = [\text{C}^+] = 3 \cdot r + r \cdot s.
\]

c) \[
(u + 1) \cdot (u - 1) = [\text{C}^+] = u^2 - 1 = [\text{C}^+] = u^2 - 1 = [\text{C}^+] = u^2 - 1.
\]

5) **Lässt sich (x+y)\cdot(x-y) = x\cdot x - y\cdot y nur mit obigen Gesetzen begründen?**

a) Wenn “ja”, geben Sie bitte die ersten drei Schritte samt Begründung an:

\[
(x+y)(x-y) = [\text{C}^+] = x \cdot x + y \cdot x - y \cdot x - y \cdot y = [\text{C}^+] = x \cdot x - y \cdot y
\]

b) Wenn “nein”, geben Sie bitte einige der fehlenden Gesetze an:
Cannot use theorems

4) **Begründen Sie bitte ebenso beim schrittweisen Vereinfachen;** machen Sie soviele Schritte wie Sie brauchen:

a) \(2 \cdot (x + 3 \cdot y) + 6 \cdot y\)
\[= A^{+} = 2 \cdot x + 12 \cdot y\]
\[= N = (2 \cdot x + 6 \cdot y) - 6 \cdot y\]
\[= A^{-} = 2 \cdot x + 6 \cdot y - 6 \cdot y\]
\[= + = 2 \cdot x + 0\]
\[= U^{+} = 2 \cdot x\]

b) \(r + r \cdot (2 + s)\)
\[= A^{+} = 2 \cdot r + r \cdot r \cdot s\]
\[= D^{+} = 2 \cdot r + r \cdot s\]
\[= N = r + r \cdot (s + r)\]
\[= D^{-} = r + r \cdot (s + r)\]

5) **Lässt sich \((x+y) \cdot (x-y) = x \cdot x - y \cdot y\)** nur mit obigen Gesetzen begründen?

a) Wenn "ja", geben Sie bitte die ersten drei Schritte samt Begründung an:
\((x+y)(x-y)\)
\[= A^{-} = \kappa^2 - \kappa \cdot y + y \cdot (x-y)\]
\[= L^{+} = \kappa^2 - \kappa \cdot y - y^2\]

b) Wenn "nein", geben Sie bitte einige der fehlenden Gesetze an:
Cannot use theorems

4) Begründen Sie bitte ebenso beim schrittweisen Vereinfachen; machen Sie so viele Schritte wie Sie brauchen:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = \]
\[ = \left[ A^+ \right] = \]
\[ = \left[ C^+ \right] = 2x + 12y \]
\[ = \left[ \ldots \right] = \]
\[ = \left[ \ldots \right] = \]
\[ = \left[ \ldots \right] = 2x + 12y. \]

b) \[ r + r \cdot (2 + s) = \]
\[ = \left[ A^+ \right] = \]
\[ = \left[ C^+ \right] = 2, 3r + 2s \]
\[ = \left[ \ldots \right] = \]
\[ = \left[ \ldots \right] = 3r + 2s. \]

c) \[ (u + 1) \cdot (u - 1) = \]
\[ = \left[ A^+ \right] = \]
\[ = \left[ C^+ \right] = \]
\[ = \left[ \ldots \right] = \]
\[ = \left[ \ldots \right] = \]
\[ = \left[ \ldots \right] = u^2 - 1. \]

5) Lässt sich \((x+y) \cdot (x-y) = x \cdot x - y \cdot y\) nur mit obigen Gesetzen begründen?

a) Wenn "ja", geben Sie bitte die ersten drei Schritte samt Begründung an:
\[ y \cdot \frac{1}{2} (x+y)(x-y) = \]
\[ = \left[ D^- \right] = \]
\[ = \left[ A^+ \right] = \]
\[ = \left[ \ldots \right] = \]
\[ = \left[ \ldots \right] = \]
\[ = \left[ \ldots \right] = \]
\[ = \left[ \ldots \right] = x^2 - y^2 = x \cdot x - y \cdot y. \]

b) Wenn "nein", geben Sie bitte einige der fehlenden Gesetze an:
Cannot use theorems

\[2 \cdot (x + 3 \cdot y) - 6 \cdot y = (2 \cdot x + 2 \cdot (3 \cdot y)) - 6 \cdot y = (2 \cdot x + (2 \cdot 3) \cdot y) - 6 \cdot y = \begin{array}{c}\text{[N]} \\
= (2 \cdot x + 6 \cdot y) - 6 \cdot y = \begin{array}{c}\text{[A+]} \\
= 2 \cdot x + (6 \cdot y - 6 \cdot y) = \begin{array}{c}\text{[I+]} \\
= 2 \cdot x + 0 = \begin{array}{c}\text{[U+]} \\
= 2 \cdot x
\end{array}
\end{array}
\end{array}\]

4) Tomando la simplificación anterior como ejemplo, simplifica las siguientes expresiones algebraicas indicando en cada paso la regla usada.

Usa tantos pasos como consideres necesarios:

a) \[2 \cdot (x + 3 \cdot y) + 6 \cdot y = \begin{array}{c}\text{[D+]} \\
= 2 \cdot x + 6 \cdot y + 6 \cdot y = \begin{array}{c}\text{[D+]} \\
= 2 \cdot x + 12 \cdot y = \begin{array}{c}\text{[D+]} \\
\end{array}
\end{array}\]

b) \[r + r \cdot (2 + s) = \begin{array}{c}\text{[D+]} \\
= r + 2 \cdot r + r \cdot s = \begin{array}{c}\text{[D+]} \\
= 3 \cdot r + r \cdot s = \begin{array}{c}\text{[D+]} \\
\end{array}
\end{array}\]

c) \[(u + 1) \cdot (u - 1) = \begin{array}{c}\text{[D+]} \\
\end{array} = u^2 - 1 = \begin{array}{c}\text{[D+]} \\
\end{array} = \begin{array}{c}\text{[D+]} \\
\end{array}\]

5) ¿Puede la transformación \((x + y) \cdot (x - y) = x^2 - y^2\) justificarse usando únicamente las reglas mencionadas más arriba?

a) Si tu respuesta es “sí”, escribe los tres primeros pasos indicando la regla usada en cada paso:

\[(x + y) \cdot (x - y) = \begin{array}{c}\text{[D+]} \\
= (x \cdot x - x \cdot y) + (y \cdot x - y \cdot y) = \begin{array}{c}\text{[I+]} \\
= x^2 - x \cdot y + x \cdot y - y^2 = \begin{array}{c}\text{[I+]} \\
\end{array}
\end{array}\]

\[= \begin{array}{c}\text{[D+]} \\
= x^2 - y^2 = \begin{array}{c}\text{[D+]} \\
\end{array} = \begin{array}{c}\text{[D+]} \\
\end{array}\]
4) Tomando la simplificación anterior como ejemplo, simplifica las siguientes expresiones algebraicas indicando en cada paso la regla usada. Usa tantos pasos como consideres necesarios:

a) \[ 2 \cdot (x + 3y) + 6y \]

\[ = [D^+] = (2x + 2(3y)) + 6y \]

\[ = [A^+] = (2x + (2 \cdot 3y)) + 6y \]

\[ = [A^+] = (2x + 6y) + 6y \]

\[ = [A^+] = 2x + 12y \]

b) \[ r + r(2 + s) \]

\[ = [D^+] = r + (2r + rs) \]

\[ = [A^+] = r + 2r + rs \]

\[ = [A^+] = r(1 + 2 + s) \]

\[ = [A^+] = r(3 + s) \]

c) \[ (u + 1) \cdot (u - 1) \]

\[ = [D^+] = (u \cdot u - u \cdot 1) + (u \cdot 1 - 1 \cdot 1) \]

\[ = [A^+] = (u^2 - u) + (u - 1) \]

\[ = [A^+] = (u^2 - u) + (u - u) \]

\[ = [A^+] = u^2 - 1 \]

5) ¿Puede la transformación \( (x + y) \cdot (x-y) = x \cdot x - y \cdot y \) justificarse usando únicamente las reglas mencionadas más arriba?

a) Si tu respuesta es “sí”, escribe los tres primeros pasos indicando la regla usada en cada paso:

\[ (x+y)(x-y) \]

\[ = [D^+] = (x \cdot x - x \cdot y) + (y \cdot x - y \cdot y) \]

\[ = [D^+] = (x^2 - xy) + (xy - y^2) \]

\[ = [D^+] = (x^2 - y^2) + (xy - xy) \]

\[ = [D^+] = x^2 - y^2 \]
4) Tomando la simplificación anterior como ejemplo, simplifica las siguientes expresiones algebraicas indicando en casa paso la regla usada.

Usa tantos pasos como consideres necesarios:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = [D+] = 2x + (2 \cdot 3 \cdot y) + 6 \cdot y \]
   \[ = [A \cdot ] = 2x + 6y + 6y \]
   \[ = [A + t] = 2x + 12y \]

b) \[ r + r \cdot (2 + s) = [D+] = r + (2 \cdot r + s \cdot r) \]
   \[ = [A \cdot ] = r + (2r + sr) \]
   \[ = [A + t] = 3r + sr \]
   \[ = [D+] = (3 + s) \cdot r \]

c) \[ (u + 1) \cdot (u - 1) = [D+] = u^2 - u + u - 1 \]
   \[ = [I \cdot ] = u^2 - 1 \]

5) ¿Puede la transformación \((x + y) \cdot (x - y) = x \cdot x - y \cdot y\) justificarse usando únicamente las reglas mencionadas más arriba?

a) Si tu respuesta es “sí”, escribe los tres primeros pasos indicando la regla usada en cada paso:

\[ (x + y)(x - y) = [D+] = x^2 - xy + xy - y^2 \]
   \[ = [I \cdot ] = x^2 - y^2 \]
Cannot use theorems

\[2(\text{x} + 3\text{y}) - 6\text{y} \equiv (2\text{x} + 2(3\text{y})) - 6\text{y} \equiv 2\text{x} + 2(3\text{y}) - 6\text{y} \equiv [N] = 2\text{x} + 6\text{y} - 6\text{y} = [\text{A+}] = 2\text{x} + (6\text{y} - 6\text{y}) = [\text{I+}] = 2\text{x} + 0 = [\text{U+}] = 2\text{x}.

4) Kao u navedenom primeru, uprostite izraze u navođenje zakona:
Uradite koliko god koraka je potrebno:
   a) \[2(\text{x} + 3\text{y}) + 6\text{y} = [\text{D+}] = (2\text{x} + 2(3\text{y})) + 6\text{y} \equiv [\text{A+}] = 2\text{x} + (2\cdot3\cdot\text{y}) + 6\text{y} \equiv [\text{I+}] = 2\text{x} + (\text{y} + 6\text{y}) = [\text{U+}] = 2\text{x} + 12\text{y}.

   b) \[\text{r} + \text{r} \cdot (2 + \text{s}) = [\text{D+}] = \text{r} + (\text{r} \cdot 2 + \text{r} \cdot \text{s}) \equiv [\text{A+}] = (\text{r} + \text{r} \cdot 2) + \text{r} \cdot \text{s} \equiv [\text{I+}] = 3\text{r} + \text{r} \cdot \text{s} = [\text{U+}] = 3\text{r} + \text{r} \cdot \text{s}.

   c) \[(\text{u} + 1) \cdot (\text{u} - 1) = [\text{D+}] = (\text{u} \cdot (\text{u} - 1) + 1 \cdot (\text{u} - 1)) \equiv [\text{A+}] = (\text{u} \cdot \text{u} - \text{u} \cdot 1 + 1 \cdot \text{u} - 1 \cdot 1) \equiv [\text{I+}] = (\text{u} \cdot \text{u} - 1 \cdot 1 + 1 \cdot \text{u} - 1 \cdot 1) = [\text{U+}] = \text{u} \cdot \text{u} - 1.

5) Da li se uprošćavanje \((\text{x} + \text{y}) \cdot (\text{x} - \text{y}) = \text{x} \cdot \text{x} - \text{y} \cdot \text{y}\) može opravdati samo navedenim zakonima?
   a) Ako mislite da može, navedite prva 3 koraka:
\[(\text{x} + \text{y}) \cdot (\text{x} - \text{y}) = [\text{D+}] = \text{x} \cdot (\text{x} - \text{y}) + \text{y} \cdot (\text{x} - \text{y}) \equiv [\text{A+}] = \text{x} \cdot \text{x} - \text{x} \cdot \text{y} + \text{y} \cdot \text{x} - \text{y} \cdot \text{y} \equiv [\text{I+}] = \text{x} \cdot \text{x} + 0 - \text{y} \cdot \text{y} = [\text{U+}] = \text{x} \cdot \text{x} - \text{y} \cdot \text{y}.\]
4) Kao u navedenom primeru, uprostite izraze u navođenje zakona:
Uradite koliko god koraka je potrebno:

a) \[2 \cdot (x + 3 \cdot y) + 6 \cdot y = \]

\[= \left( 2 \cdot x + (2 \cdot 3) \cdot y \right) + 6 \cdot y = \]

\[= \left( 2 \cdot x + 6 \cdot y \right) + 6 \cdot y = \]

\[= 2 \cdot x + (6 \cdot y - 6 \cdot y) = \]

\[= 2 \cdot x + 0 = \]

\[= 2 \cdot x \]

b) \[r \cdot (r + (2 \cdot s)) = \]

\[= r + \left( 2 \cdot r + r \cdot s \right) = \]

\[= r + 2 \cdot r + r \cdot s = \]

\[= r \left( 1 + 2 + s \right) = \]

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Cannot use theorems

\[2(x + 3 \cdot y) + 6 \cdot y = (2 \cdot x + 6 \cdot y) - 6 \cdot y = 2 \cdot x + (6 \cdot y - 6 \cdot y) = 2 \cdot x + 0 = 2 \cdot x\]

4) Kao u navedenom primjeru, uprostite izraze u navođenje zakona:
Uradite koliko god koraka je potrebno:

a) \[2 \cdot (x + 3 \cdot y) + 6 \cdot y = (2 \cdot x + 2 \cdot 3 \cdot y) + 6 \cdot y = (2 \cdot x + 3 \cdot (2 \cdot y)) + 6 \cdot y = (2 \cdot x + 6 \cdot y) + 6 \cdot y = 2 \cdot x + 12 \cdot y\]

b) \[r + r \cdot (2 + s) = (r + r \cdot 2) \cdot s = (r + 2 \cdot r) \cdot s = (r + 2 \cdot r) \cdot s = (r + 2 \cdot r) \cdot s = (r + 2 \cdot r) \cdot s = (r + 2 \cdot r) \cdot s\]

c) \[(u + 1) \cdot (u - 1) = (u + 1 \cdot u - u - 1) = (u + u - 1 - 1) = (2 \cdot u - 2) = 2 \cdot u - 2\]

5) Da li se upроšćavanje \((x+y)(x-y) = x \cdot x - y \cdot y\) može opravdati samo navedenim zakonima?

a) Ako mislite da može, navedite prva 3 koraka:

\[(x+y)(x-y) = (x \cdot x - y \cdot x) + (x - y) = (x^2 - yx) + (y - x^2) = (x \cdot x - y \cdot y) = x \cdot x - y \cdot y\]
Cannot use theorems

\[ 2 \cdot (x + 3 \cdot y) - 6 \cdot y =^{(\text{Left Distrib.})} (2 \cdot x + 2 \cdot (3 \cdot y)) - 6 \cdot y =^{(\text{Addition})} (2 \cdot x + (2 \cdot 3) \cdot y) - 6 \cdot y =^{(\text{Distrib.})} (2 \cdot x + 6 \cdot y) - 6 \cdot y =^{(\text{Subtraction})} 2 \cdot x + (6 \cdot y - 6 \cdot y) =^{(\text{Subtraction})} 2 \cdot x + 0 =^{(\text{Identity})} 2 \cdot x \]

4) Similarly describe a stepwise justification of the following simplifications:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y =^{(\text{Distrib.})} 2 \cdot x + 2 \cdot 3 \cdot y + 6 \cdot y =^{(\text{Distrib.})} 2 \cdot x + 6 \cdot y + 6 \cdot y =^{(\text{Addition})} 2 \cdot x + 6 \cdot (y + y) =^{(\text{Distrib.})} 2 \cdot x + 6 \cdot y + 6 \cdot y \]

b) \[ r + r \cdot (2 + s) =^{(\text{Distrib.})} r + (2 \cdot r + r \cdot s) =^{(\text{Distrib.})} 3 \cdot r + r \cdot s \]

c) \[ (u + 1) \cdot (u - 1) =^{(\text{Distrib.})} u \cdot u - u + u - 1 =^{(\text{Subtraction})} u^2 - u + u - 1 =^{(\text{Subtraction})} u^2 - 1 \]

5) Can the simplification \((x+y)\cdot(x-z) = x \cdot x - y \cdot y\) be justified using the above laws only?

a) If “yes”, give the first three steps and justifications:

\[ (x+y)(x-z) =^{(\text{Distrib.})} x \cdot x - x \cdot z + y \cdot x - y \cdot z =^{(\text{Distrib.})} x \cdot x - x \cdot y + y \cdot y - y \cdot z \]

b) If “no”, give some missing laws:
Cannot use theorems

\[ 2 \cdot (x + 3 \cdot y) - 6 \cdot y \overset{[N]}{=} (2 \cdot x + 2 \cdot (3 \cdot y)) - 6 \cdot y \overset{[A++]}{=} (2 \cdot x + (2 \cdot 3) \cdot y) - 6 \cdot y \overset{[I^+]}{=} 2 \cdot x + (6 \cdot y - 6 \cdot y) \overset{[U^+]}{=} 2 \cdot x + 0 = 2 \cdot x \]

4) Similarly describe a stepwise justification of the following simplifications:
   a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = \ldots = (2 \cdot x + 3 \cdot y) + 6 \cdot y = \ldots = 2 \cdot x + 3 \cdot y + 6 \cdot y \]
   b) \[ r + r \cdot (2 + s) = \ldots = r + (2 \cdot r + rs) = \ldots = r + 2 \cdot r + rs \]
   c) \[ (u + 1) \cdot (u - 1) = \ldots = u^2 - 1 \cdot n + 1 \cdot n - 1 = \ldots = u^2 - n + n - 1 \]

5) Can the simplification \((x+y) \cdot (x-z) = x \cdot x - y \cdot y\) be justified using the above laws only?
   a) If “yes”, give the first three steps and justifications:
   \[ (x+y)(x-z) = \ldots = \ldots = \ldots = x \cdot x - y \cdot y \]
   b) If “no”, give some missing laws:
Cannot use theorems

\[ (x + y) - 6 \cdot y = (2 \cdot x + 6 \cdot y) - 6 \cdot y = 2 \cdot x + (6 \cdot y - 6 \cdot y) = 2 \cdot x + 0 = 2 \cdot x \]

4) Similarly describe a stepwise justification of the following simplifications:
   a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = 2x + 6y + 6y = 2x + 12y = x + 6y \]
   b) \[ r + r \cdot (2 + s) = r + 2r + rs = 3r + rs \]
   c) \[ (u + 1) \cdot (u - 1) = u^2 - u + u - 1 = u^2 - 1 \]

5) Can the simplification \((x+y) \cdot (x-1) = x \cdot x - y \cdot y\) be justified using the above laws only?
   a) If “yes”, give the first three steps and justifications:
      \[ (x+y)(x-1) = x^2 - x + y - y = x^2 - x \]
   b) If “no”, give some missing laws:
Cannot use theorems

4) Kao u navedenom primeru, uprostite izraze u navođenje zakona:
Uradite koliko god koraka je potrebno:

a) $2 \cdot (x + 3 \cdot y) + 6 \cdot y = (2 \cdot x + 2 \cdot (3 \cdot y)) + 6 \cdot y = (2 \cdot x + (2 \cdot 3 \cdot y)) + 6 \cdot y$

$= [[N]] = (2 \cdot x + 6 \cdot y) - 6 \cdot y = [[A+]] = 2 \cdot x + (6 \cdot y - 6 \cdot y) = [[U]] = 2 \cdot x + 0 = [[U]] = 2 \cdot x$

b) $r \cdot r \cdot (2 + s) = (r \cdot (2 \cdot s)) + r \cdot s$

$= [[A+H]] = (2 \cdot s \cdot r) + r \cdot s$

$= [[U]] = 2 \cdot r \cdot (3 + s) = [[U]]$

5) Da li se uprošćavanje $(x+y) \cdot (x-y) = x \cdot x - y \cdot y$ može opravdati samo navedenim zakonima?

a) Ako mislite da može, navedite prva 3 koraka:

$(x+y)(x-y) = (x+y) \cdot (x-y) = (x+y) \cdot x - (x+y) \cdot y$

$= [[A+H]] = (x \cdot x + x \cdot y) - (y \cdot x + y \cdot y)$

$= [[U]] = x \cdot x - y \cdot y$
Cannot use theorems

\[
2 \cdot (x + 3 \cdot y) - 6 \cdot y \quad \Rightarrow \quad (2 \cdot x + 2 \cdot (3 \cdot y)) - 6 \cdot y \\
= [N] = (2 \cdot x + 6 \cdot y) - 6 \cdot y \quad = [A+] = 2 \cdot x + 6 \cdot y - 6 \cdot y \\
= [I^+] = 2 \cdot x + 0 = [U^+] = 2 \cdot x
\]

4) Tomando la simplificación anterior como ejemplo, simplifica las siguientes expresiones algebraicas indicando en cada paso la regla usada.
Usa tantos pasos como consideres necesarios:

a) \[2 \cdot (x + 3 \cdot y) + 6 \cdot y = [D+] = (2 \cdot x + 6 \cdot y) + 6 \cdot y \]
\[= [A++] = 2 \cdot x + 6 \cdot y + 6 \cdot y \]
\[= [C+] = 2 \cdot x + 12 \cdot y \]
\[= [U_] = 2 \cdot x + 12 \cdot y \]

b) \[r + r \cdot (2 + s) = [D+] = r + (2 \cdot r + r \cdot s) \]
\[= [A++] = (r + 2 \cdot r) + r \cdot s \]
\[= [C+] = 3 \cdot r + r \cdot s \]
\[= [D+] = r (3 + s) \]

c) \[(u + 1) \cdot (u - 1) = [D+] = u (u - 1) + 1 \cdot (u - 1) \]
\[= [D^-] = u^2 - u + 1 \cdot (u - 1) \]
\[= [D^-] = u^2 - u + u - 1 \]
\[= [I+] = u^2 - 1 \]

5) ¿Puede la transformación \((x + y) \cdot (x - y) = x \cdot x - y \cdot y\) justificarse usando únicamente las reglas mencionadas más arriba?

a) Si tu respuesta es “sí”, escribe los tres primeros pasos indicando la regla usada en cada paso:

\[(x+y)(x-y) = [D+] = x \cdot (x+y) - y \cdot (x+y) \]
\[= [D+] = x^2 + x \cdot y - x \cdot y - y^2 \]
\[= [I+] = x^2 - y^2 \]
\[= x \cdot x - y \cdot y \]
Cannot use theorems

\[ 2 \cdot (x + 3 \cdot y) - 0 \cdot y \]

\[= [N^-] (2 \cdot x + 6 \cdot y) - 6 \cdot y \]

\[= [A^-] 2 \cdot x + (6 \cdot y - 6 \cdot y) \]

\[= [I^+] 2 \cdot x + 0 \]

\[= [U^+] 2 \cdot x \]

4) **Begründen Sie bitte ebenso beim schrittweisen Vereinfachen**; machen Sie so vieler Schritte wie Sie brauchen:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y \]

\[= [D^+] (2 \cdot x + 2 \cdot (3 \cdot y)) + 6 \cdot y \]

\[= [A^-] (2 \cdot x + (2 \cdot 3) \cdot y) + 6 \cdot y \]

\[= [N^-] 2 \cdot x + 6 \cdot y + 6 \cdot y \]

\[= [I^+] 2 \cdot x + 6 \cdot y \]

\[= [U^+] 2 \cdot x + 12 \cdot y \]

b) \[ r + r \cdot (2 + s) \]

\[= [D^+] r + 2r + r \cdot s \]

\[= [A^-] \]

\[= [N^-] \]

\[= [I^+] r + 2r + r \cdot s \]

\[= [U^+] 3r + r \cdot s \]

c) \[ (u + 1) \cdot (u - 1) \]

\[= [D^-] (u + 1) \cdot u - (u + 1) \cdot 1 \]

\[= [D^+] u^2 + u - u - 1 \cdot 1 \]

\[= [A^-] \]

\[= [N^-] \]

\[= [I^+] u^2 - 1 \]

5) **Lässt sich** \((x+y) \cdot (x-y) = x \cdot x - y \cdot y \) **nur mit obigen Gesetzen begründen?**

a) Wenn "ja", geben Sie bitte die ersten drei Schritte samt Begründung an:

\[ (x+y) (x-y) \]

\[= [D^-] (x+y) \cdot x - (x+y) \cdot y \]

\[= [B^-] \]

\[= [D^+] x \cdot x + y \cdot x - x \cdot y - y \cdot y \]

\[= [N^+] x \cdot x - y \cdot y \]

b) Wenn "nein", geben Sie bitte einige der fehlenden Gesetze an:
Cannot use theorems

4) Similarly describe a stepwise justification of the following simplifications:

a) \(2 \cdot (x + 3 \cdot y) + 6 \cdot y = [D+^+]= 2 \cdot x + 2 \cdot (3 \cdot y) + 6 \cdot y = [A^+] = 2 \cdot x + (2 \cdot 3) \cdot y + 6 \cdot y = [D^+]= 2 \cdot x + 6 \cdot y + 6 \cdot y = [..]= 2 \cdot x + 6 \cdot y + 6 \cdot y = [D^+]= 2 \cdot x + (6 \cdot 6 + 6) \cdot y = 2 \cdot x + 12 \cdot y = [..]=

b) \(r + r \cdot (2 + s) = [D^+] = r \cdot (1 + (2 + s)) = [A^+] = r \cdot ((1 + 2) + s) = r \cdot (3 + s) = [..]=

c) \((u + 1) \cdot (u - 1) = [D^+] = (u + 1) \cdot u - (u + 1) \cdot 1 = [P+] = u^2 + u - (u + 1) = [..]= u^2 + u - u - 1 = [A^+] = u^2 + (u - u) = 0 = [D^+] = u^2 - 1 = [..]=

5) Can the simplification \((x + y) \cdot (x + y) = x \cdot x - y \cdot y\) be justified using the above laws only?

a) If “yes”, give the first three steps and justifications:
\((x + y)(x + y) = [D^-]= (x + y) \cdot x - (x + y) \cdot y = [D^+]= x^2 + y \cdot x - (x \cdot y + y^2) = [..]= x^2 + y \cdot x + x^2 = [A^+]= x^2 + (x \cdot x - y \cdot y) = x^2 + x \cdot x - y \cdot y = x^2 + x^2 - y^2 = [..]= x^2 + (x \cdot x - y \cdot y) + y^2 = x^2 + (x \cdot x) + y^2 = x^2 + y^2 = [..]=

b) If “no”, give some missing laws:
Summary, Interpretation?

1. Front-side: students can simplify
   - They perfectly operate on formulas
   - Do they have a notion of 'theorem'?

2. Front-side: students know theorems
   - They demonstrate comprehension of “theorem/rule”: one rule for all numbers
   - All great !!!!!?

3. Back-side: students cannot use theorems (…proving!)

Interpretation?
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Interpretation?

(1)(2) object language: formulas address “real world”
(3) meta language: formulas address/prove other formulas

Self-referentiality in language is not a recognised difficulty! (... by didactics, but clearly implemented in TP-based systems)!
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FM: Demand for Proving

- “Formal Methods” (FM) — an advancing discipline:
  - increases demand for mathematics in engineering:
    1. FM extends the field of application of mathematics
    2. FM explicitly specify systems’ features (‘formalise’)
    3. FM verify/prove properties of SW/HW components
    4. . . thus increases the demand for proving.

**definition segment :: "Point ⇒ Point ⇒ Point set"**
**where "segment x y =**

{z. ∃t. 0≤t ∧ t≤1 ∧ (z = t *ₚ x + (1-t) *ₚ y )}"

**definition is_convex :: "Point set ⇒ bool"**
**where "is_convex K = (∀x∈K. ∀y∈K. segment x y ⊆ K)"**
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\[
\text{definition segment :: "Point } \Rightarrow \text{ Point } \Rightarrow \text{ Point set"}
\]
\[
\text{where } \ "\text{segment } x \ y = \{z. \ \exists t . \ 0 \leq t \land t \leq 1 \land (z = t \ast_R x + (1-t) \ast_R y) \}\"
\]

\[
\text{definition is_convex :: "Point set } \Rightarrow \text{ bool"}
\]
\[
\text{where } \ "\text{is_convex } K = (\forall x \in K. \ \forall y \in K. \ \text{segment } x \ y \subseteq K)"
\]
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Fig. 5 Postprocessing step 3: Convert safety zone from internal Sphere Swept Convex Hull (SSCH) representation into a laser scan like representation. In this representation the safety zone can be safeguarded by simply comparing with a laser scan.
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\[ T(s, \alpha). \] This results in the overall computation
\[
H(s_{\text{min}}, s_{\text{max}}, \alpha_{\text{min}}, \alpha_{\text{max}}) = A \left( \left[ \left[ P_{i,s,\alpha}^{1}, P_{i,s,\alpha}^{2}, \left[ V_{i,s,\alpha}^{j} \right]_{j=0}^{L-1} \right]_{i=0}^{n} \right] s_{\text{min}} \right) s_{\text{max}} \alpha_{\text{min}} \alpha_{\text{max}} : q, \]

with
\[
q = q^A + q^B
\]
\[
q^A = \frac{1}{6} \left( \frac{\alpha_{\text{max}} - \alpha_{\text{min}}}{2} \right)^2 \max \{ |s_{\text{max}}|, |s_{\text{min}}| \}
\]
\[
q^B = (1 - \cos \frac{\alpha_{\text{max}} - \alpha_{\text{min}}}{2}) \max \{ |R_i| \}.
\]

4. ... thus increases the demand for proving.
“Formal Methods” (FM) — an advancing discipline:

- increases demand for proofs in engineering:
  1. FM extends the field of application of mathematics
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  4. . . thus increases the demand for proving.

Fig. 1 The SAMS demonstrator driving a right hand bent and the collision-free safety zone of that movement. If there was any obstacle inside the safety zone the AGV would stop.
FM: Demand for Proving

Can we address specific difficulties in teaching to prove before we start teaching to prove?

Can we develop specific abilities with the students as prerequisites for proving?

Can we have a continuum from high-school math to academic math (involving proofs)?
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Outline
GeoGebra’s “academic relative”, see `prove {identical O_1 O_2}` at bottom left.
"Structured Derivations", Turku

Structured derivation editor with erroneous step marked.
See E-Math Project http://emath.eu/
For *step-wise problem solving* in applied mathematics. Realises the **three features** expected for TP-based systems.
For software “correct by construction” in education.
Conceptual Foundations

Mathematics is the science of reasoning . . .

- each fact/operation can be proved
- . . . “proving” is the essence which distinguishes math.

TP ((Computer) Theorem Proving) implements this essence to an extent . . .

- . . . such that TP-based systems can be conceived as “models of mathematics”, which
  - allow learning by interaction with these models

So the primary focus of development is

- clear logical foundations for the system
- (and not the structure of learning, i.e. Artificial Intelligence, Cognitive Science, etc).
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Features for Interactivity

These features arise from TP . . .

1. check user input automatically, **flexibly** and reliably: Input establishes a *proof situation* (for *automated* proving) with respect to the logical context

2. give explanations on request by learners: All underlying mathematics knowledge is **transparent** due to the “LCF-paradigm” in Isabelle (standard predicate logic)

3. propose a next step if learners get stuck: “next-step-guidance” due to Lucas-Interpretation.

They feature software support for:

- **step-wise solving math** problems in STEM ¹
- learning interactively like with a **chess-program**
- . . .

---

¹“STEM” is **Science, Technology, Engineering, and Mathematics**.
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They feature software support for:

- **step-wise solving math** problems in STEM \(^1\)
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\(^1\)“STEM” is *Science, Technology, Engineering and Mathematics*.
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Thank you for Attention!
P. Quaresma and R.-J. Back, editors,
Proceedings First Workshop on *TP Components for Educational Software*.

Theorem Prover Isabelle, Library of Math Knowledge
http://isabelle.in.tum.de/dist/library/HOL/index.html

E-Math Project: Improving Competence in Mathematics using New Teaching Methods and ICT
http://emath.eu/