Irreplacable Contributions of Math Assistants to Learning
On Psychology of “Convergence on Math Assistants”

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Special Track “Convergence on Mathematics Assistants”
Outline

1. Mathematics and language

2. An EU-wide questionnaire
   Organisational framework
   Front-side: students can simplify
   Front-side: students know theorems
   Back-side: students cannot use theorems

3. Consequences . . .
   . . . for didactics research
   . . . for R&D on math assistants

4. Summary
Mathematics and language

Two kinds (stages ?) of language:

1. Natural language, the “mother tongue”
   - learned by imitation | adjustment to good / bad
   - concerns emotion | creativity | communication
   - develops naturally alongside mental maturing

2. The formal language of mathematics
   - each step justified (finally by formal logics)
   - formality ensures reliability
   - formulas scale up into infinite complexity

When get students familiar with formal language ?
How get students familiar with formal language ?
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When get students familiar with formal language?
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Irreplacable Role of Math. Assistants
Walther Neuper

Mathematics – language

Querstionnaire Framework
Simplify Know theorems
Use theorems

Consequences Psychology
Techn. R&D

Summpary

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4 Summary
An EU-wide questionnaire

- distributed and collected in spring 2010
- in Austria, Finland, Romania, Serbia, Spain
- at Universities, engineering courses
- ≈ 25 minutes duration
- ≈ 250 questionnaires
- improved questioning planned.
Mathematics – language

Questionnaire

Framework

Simplify

Know theorems

Use theorems

Consequences

Psychology

Techn. R&D

Summary

Irreplacable Role of Math. Assistants

Walther Neuper

Theorems justifying steps in algebraic transformations

University of ……………………. (if possible, change icon accordingly)

Mathematics is considered a difficult subject. This questionnaire is part of a research on basic principles and reasons for difficulties in mathematics education. Thus you are just asked basic knowledge, most of which will remind you of early mathematics classes.

1) Below you find basic laws of algebra (i.e. theorems). (a) Do you remember some laws? (b) Do you even remember the names of some laws? (c) Can you apply the laws to numbers?

a) \( a + b = b + a \) (x yes ☐ no) (b) law of commutativity for +

b) \( a \cdot b = b \cdot a \) ☐ yes ☐ no law ________________

c) \( (a + b) + c = a + (b + c) \) ☐ yes ☐ no law ________________

d) \( a \cdot (b + c) = a \cdot b + a \cdot c \) ☐ yes ☐ no law ________________

e) \( a + b = b + a \) ☐ yes ☐ no law ________________

2) Do you remember any other mathematical laws, not yet mentioned above?

a) ________________ law ________________

b) ________________ law ________________

c) ________________ law ________________

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra, but usually one simplifies without laws, for instance:

a) \( 2 \cdot (x + 3) - 6y - 2 \cdot x + 6y = 2x \).

b) \( 2 \cdot (x + 3) + 6y = \) ________________

c) \( r + r(2 + s) = \) ________________

d) \( (u + 1) \cdot (u - 1) = \) ________________

e) \( (x + y) \cdot (x - y) = \) ________________

4) Similarly describe a stepwise justification of the following simplifications, please; Take as many steps you need:

a) \( 2 \cdot (x + 3) + 6y = \) ________________

b) \( r + r(2 + s) = \) ________________

c) \( (u + 1)(u - 1) = \) ________________

5) Can the simplification \( (r+s)(x-y) = x \cdot r - y \cdot s \) be justified using the above laws only?

a) If "yes", give the first three steps and justifications, please:

\( (x+y)(x-y) = \) ________________

b) If "no", give some missing laws, please:

\( (x+y)(x-y) = \) ________________

6) I am a student in a degree course for mathematics (at university) ☐ yes ☐ no

Thank you for your attention!

For results see http://www.ist.tugraz.at/projects/issac/www/content/status.html?quest10
Theorems justifying steps in algebraic transformations

University of ____________________ (if possible, change icon accordingly)

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1) Below you find basic laws of algebra (i.e. theorems). (a) Do you remember some laws? (b) Do you even remember the names of some laws? (c) Can you apply the laws to numbers?

   a) \( a + b = b + a \) \( \text{X yes } \checkmark \text{ no} \) (b) law of commutativity for +
   (c) \( 2 + 3 = 3 + 2 \) \( 5 = 5 \)

   b) \( a \cdot b = b \cdot a \) \( \text{X yes } \checkmark \text{ no} \) law ________________

   c) \( (a + b) + c = a + (b + c) \) \( \text{X yes } \checkmark \text{ no} \) law ________________

   d) \( a \cdot (b + c) = a \cdot b + a \cdot c \) \( \text{X yes } \checkmark \text{ no} \) law ________________

   e) \( a \cdot 1 = a \) \( \text{X yes } \checkmark \text{ no} \) law ________________

   f) \( a + 0 = a \) \( \text{X yes } \checkmark \text{ no} \) law ________________

   g) \( a - a = 0 \) \( \text{X yes } \checkmark \text{ no} \) law ________________

2) Do you remember any other mathematical laws, not yet mentioned above?

   a) ________________ law ________________

   b) ________________ law ________________

   c) ________________ law ________________

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra, but usually one simplifies without laws, for instance:

   a) \( 2(3x - 3y) - 6y = 2x + 6y - 6y = 2x \)

   b) \( 2x + 3y + 3y = x + 7y \)

   c) \( r + r(2 + s) = r + 2r + rs = 3r + rs \)

   d) \( (a + 1) \cdot (a - 1) = a^2 - 1 \)

   e) \( (x + y) \cdot (x - y) = x^2 - y^2 \)

4) Similarly describe a stepwise justification of the following simplifications, please; Take as many steps you need:

   a) \( 2(x + 3y) + 6y = 2x + 3y + 6y \)

   b) \( r + r(2 + s) \)

   c) \( (a + 1)(a - 1) \)

5) Can the simplification \( (x + y)(x - y) = x^2 - y^2 \) be justified using the above laws only?

   a) If “yes”, give the first three steps and justifications, please:

      \( (x+y)(x-y) = x^2 - y^2 \)

   b) If “no”, give some missing laws, please:

   c) law ________________

6) I am a student in a degree course for mathematics (at university) \( \text{X yes } \checkmark \text{ no} \)

Thank you for your attention!

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4 Summary
### Theorems justifying steps in algebraic transformations

University of ………………………… (if possible, change icon accordingly)

Mathematics is considered a difficult subject. This questionnaire is part of a research on basic principles and reasons for difficulties in mathematics education. Thus you are just asked basic knowledge, most of which will remind you of early mathematics classes.

1) Below you find basic laws of algebra (i.e. theorems). (a) Do you remember some laws? (b) Do you even remember the names of some laws? (c) Can you apply the laws to numbers?

| a + b = b + a | (a) X yes ☐ no | (b) law of commutativity for + |
| 2 + 3 = 3 + 2 | … 5 = 5 |

| a · b = b · a | (a) X yes ☐ no | (b) law of commutativity for ⋅ |
| (a + b) + c = a + (b + c) | (a) X yes ☐ no | (b) law of associativity |
| a · b · c = (a · b) · c | (a) X yes ☐ no | (b) law of associativity |

2) Do you remember any other mathematical laws, not yet mentioned above?

| a · b = b · a | (a) X yes ☐ no | (b) law of commutativity for ⋅ |
| 2 + 3 = 3 + 2 | … 5 = 5 |

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra, but usually one simplifies without laws, for instance:

| a + b = b + a | (a) X yes ☐ no | (b) law of commutativity for + |
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| a + b = b + a | (a) X yes ☐ no | (b) law of commutativity for + |
| 2 + 3 = 3 + 2 | … 5 = 5 |

5) Can the simplification (x + y)(x – y) = x² – y² be justified using the above laws only?

| a + b = b + a | (a) X yes ☐ no | (b) law of commutativity for + |
| 2 + 3 = 3 + 2 | … 5 = 5 |

6) I am a student in a degree course for mathematics (at university) ☐ yes ☐ no

Thank you for your attention!

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Students can simplify

\[ a - a = 0 \]
\[ \square \text{yes} \square \text{no} \]
law \[ \quad \]

2) Do you remember any other mathematical laws, not yet mentioned above?

a) \[ \quad \]
law \[ \quad \]
b) \[ \quad \]
law \[ \quad \]
c) \[ \quad \]
law \[ \quad \]

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra; but usually one simplifies without laws, for instance:

a) \[ 2 \cdot (x + 3 \cdot y) - 6y = 2 \cdot x + 6 \cdot y - 6 \cdot y = 2 \cdot x \]
b) \[ 2 \cdot (x + 3 \cdot y) + 6y = \quad = \]
c) \[ r + r \cdot (2 + s) = \quad = \]
d) \[ (u + 1) \cdot (u - 1) = \quad = \]
e) \[ (x + y) \cdot (x - y) = \quad = \]
2) Erinnern Sie sich auch an andere Gesetze (die Sie nicht schon oberhalb finden)?
   a) ................................................................. Gesetz .................................................................
   b) ................................................................. Gesetz .................................................................
   c) ................................................................. Gesetz .................................................................

3) Vereinchen Sie bitte die folgenden Ausdrücke! Rechensätze werden zusammen mit solchen Vereinfachungen gelernt; diese gelingen auch ohne Rechensätze richtig, z.B.:
   a) \(2 \cdot (x + 3 \cdot y) - 6y = 2 \cdot x + 6 \cdot y - 6 \cdot y = 2 \cdot x\)
   b) \(2 \cdot (x + 3 \cdot y) + 6y = 2x + 6y + 6y = 2x + 12y\)
   c) \(r + r \cdot (2 + s) = r + 2r + rs\)
   d) \((u + 1) \cdot (u - 1) = u^2 + u - u - 1\)
   e) \((x + y) \cdot (x - y) = x^2 + xy - xy - y^2\)
2) ¿Te acuerdas de alguna regla matemática no mencionada en el apartado anterior?
   a) el elemento inverso
   b) valor absoluto
   c) diferencia de cuadrados

3) Simplifica las siguientes expresiones algebraicas. La simplificación de expresiones se aprende habitualmente junto con las reglas algebraicas, pero por lo general tendemos a simplificar sin pensar en las reglas, por ejemplo:
   a) \(2 \cdot (x + 3 \cdot y) - 6y = 2x + 6y - 6y = 2x\)
   b) \(2 \cdot (x + 3 \cdot y) + 6y = \frac{2x + 6y + 6y}{2} = \frac{2x + 6y}{2}\)
   c) \(r + r(2 + s) = r + 2r + rs\)
   d) \((u + 1) \cdot (u - 1) = u^2 - u + u - 1^2 = u^2 - 1^2\)
   e) \((x + y) \cdot (x - y) = x^2 - xy + xy - y^2 = x^2 - y^2\)
Students can simplify

1) \( a - a = 0 \) (da) ne

\( 5 - 5 = 0 \) \( \bigcirc = \bigcirc \)

2) Možete li navesti još neke zakone koji nisu gore navedni?
   a) 
   b) 
   c) 

   zakon ________________________.
   zakon ________________________.
   zakon ________________________.

3) Uporabite naredne izraze:

   Primer: \( 2 \cdot (x + 3 \cdot y) - 6y = 2 \cdot x + 6 \cdot y - 6 \cdot y = 2 \cdot x \).
   a) \( 2 \cdot (x + 3 \cdot y) + 6y = \frac{2x + 6y + 6y}{6y} = \frac{2x + 12y}{6y} = \frac{2x + 12y}{6y} \)
   b) \( r + r(2 + s) = \frac{r + 2r + 2s}{2} = \frac{3r + 2s}{2} \)
   c) \( (u + 1) \cdot (u - 1) = \frac{2u u u + u - 1}{u u - 1} = \frac{2u}{u - 1} \)
   d) \( (x + y) \cdot (x - y) = \frac{xy y x - y y}{x y y y} = \frac{x^2 - y^2}{x y y y} \)
Students can simplify

2) Which laws (theorems) not yet mentioned above do you remember?
   a) .......................................................... law ..................................................
   b) .......................................................... law ..................................................
   c) .......................................................... law ..................................................

3) Algebraic laws are learned together with transformations generally done without using laws, for instance:
   a) \(2 \cdot (x + 3 \cdot y) - 6y = 2 \cdot x + 6 \cdot y - 6 \cdot y = 2 \cdot x\)
   b) \(2 \cdot (x + 3 \cdot y) + 6y = \frac{2 \cdot x + 6y \cdot 1 \cdot 6 \cdot y}{1 \cdot 1 \cdot 1} = \frac{2x + 12y}{1}\)
   c) \(r + r \cdot (2 + s) = r + 2.1 \cdot r + s = \frac{r + 2.1 \cdot r + s}{1/1} = \frac{3r + s}{1}\)
   d) \((u + 1) \cdot (u - 1) = \frac{(u + 1) \cdot (u - 1)}{1/1} = \frac{u^2 + u - u - 1}{1} = \frac{u^2 - 1}{1}\)
   e) \((x + y) \cdot (x - y) = \frac{x \cdot (x - y) \cdot y \cdot (x - y)}{1/1} = \frac{x^2 - xy + xy - y^2}{1} = \frac{x^2 - y^2}{1}\)
Students can simplify

- Front-side: students can simplify
  - They perfectly operate on formulas
  - Do they have a notion of theorem?
  - I.e. do they comprehend formal language?
Students can simplify

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• Front-side: students can simplify
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Students *can* simplify

- Front-side: students *can* simplify
  - They **perfectly operate** on *formulas*
  - Do they have a notion of **theorem**?
  - I.e. do they **comprehend** *formal* language?
Students can simplify

- Front-side: students can simplify
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Irreplacable Role of Math. Assistants
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Mathematics – language

Querstionnaire

Framework

Simplify

Know theorems

Use theorems

Consequences

Psychology

Techn. R&D

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Theorems justifying steps in algebraic transformations

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   a) \( a + b = b + a \) (\( \checkmark \) yes \( \Box \) no) (b) law of commutativity for +
   b) \( a \cdot b = b \cdot a \) (\( \checkmark \) yes \( \Box \) no) (c) law _________________________
   c) \((a + b) + c = a + (b + c)\) (\( \checkmark \) yes \( \Box \) no) (d) law _________________________
   d) \( a(b + c) = ab + ac \) (\( \checkmark \) yes \( \Box \) no) (e) law _________________________
   e) \( a + 0 = a \) (\( \checkmark \) yes \( \Box \) no) (f) law _________________________
   f) \( a - 0 = a \) (\( \checkmark \) yes \( \Box \) no) (g) law _________________________

   2) Do you remember any other mathematical laws, not yet mentioned above?
   a) _________________________ law _________________________
   b) _________________________ law _________________________
   c) _________________________ law _________________________

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra, but usually one simplifies without laws, for instance:

   a) \( 2(x + 3) - 6y - 2x + 6y = \) _________________________
   b) \( 2(x + 3) + 6y = \) _________________________
   c) \( r + r(2 + s) = \) _________________________
   d) \( (u + 1) \cdot (u - 1) = \) _________________________
   e) \( (x + y) \cdot (x - y) = \) _________________________

4) Similarly describe a stepwise justification of the following simplifications, please; take as many steps you need:

   a) \( 2(x + 3) + 6y - 2x + 6y = \) _________________________
   b) \( r + r(2 + s) = \) _________________________
   c) \( (u + 1) \cdot (u - 1) = \) _________________________

5) Can the simplification \((x + y)(x - y) = x \cdot x - y \cdot y\) be justified using the above laws only?

   a) If “yes”, give the first three steps and justifications, please:

   \( (x+y)(x-y) = \) _________________________

   b) If “no”, give some missing laws, please:

   law _________________________
   law _________________________
   law _________________________

6) I am a student in a degree course for mathematics (at university) (\( \checkmark \) yes \( \Box \) no)

Thank you for your attention!

For results see http://www.ist.tugraz.at/projects/issac/www/content/status.html#quest10
Students know theorems

Knowledge, most of which will remind you of early mathematics classes.

1) Below you find basic laws of algebra (i.e., theorems). (a) Do you remember some laws? (b) Do you even remember the names of some laws? (c) Can you apply the laws to numbers?

   a) $a + b = b + a$  
      (a) X yes □ no  
      (b) law of commutativity for +
      (c) $2 + 3 = 3 + 2$ ... $5 = 5$

   b) $a \cdot b = b \cdot a$
      □ yes □ no  
      law ____________________________
      .... .... = .... ....  ... ... = ....

   c) $(a + b) + c = a + (b + c)$
      □ yes □ no  
      law ____________________________
      .... .... = .... ....  ... ... = ....

   d) $a \cdot (b + c) = a \cdot b + a \cdot c$
      □ yes □ no  
      law ____________________________
      .... .... = .... ....  ... ... = ....

   e) $a \cdot 1 = a$
      □ yes □ no  
      law ____________________________
      .... .... = .... ....  ... ... = ....

   f) $a + 0 = a$
      □ yes □ no  
      law ____________________________
      .... .... = .... ....  ... ... = ....

   g) $a - a = 0$
      □ yes □ no  
      law ____________________________
      .... .... = .... ....  ... ... = ....
Students know theorems

Knowledge, most of which will remind you of early mathematics classes.

1) Below you find basic laws of arithmetic (i.e. theorems). Do you know any of these laws? Do you even remember the names of some laws? Can you apply the laws to numbers?

a) \(a + b = b + a\)  
   \[2 + 3 = 3 + 2 \quad \ldots \quad 5 = 5\]  
   \(\checkmark\) yes \(\square\) no  
   law of commutativity for +

b) \(a \cdot b = b \cdot a\)  
   \[2 \cdot 3 = 3 \cdot 2 \quad \ldots \quad 6 = 6\]  
   \(\checkmark\) yes \(\square\) no  
   law

\(\text{Kommutativ\-}yes\ +\)

(c) \((a + b) + c = a + (b + c)\)  
   \((1+2)+3 = 1+(2+3)\)  
   \[3 + 3 = 1 + 5 \quad \ldots \quad 6 = 6\]  
   \(\checkmark\) yes \(\square\) no  
   law

\(\text{Asozial\-}yes\ +\)

d) \(a \cdot (b + c) = a \cdot b + a \cdot c\)  
   \(1 \cdot (2 + 3) = 1 \cdot 2 + 1 \cdot 3\)  
   \[\checkmark\] yes \(\square\) no  
   law

\(\text{Distributiv\-}yes\ +\)

e) \(a \cdot 1 = a\)  
   \[5 \cdot 1 = 5 \quad \ldots \quad 5 = 5\]  
   \(\checkmark\) yes \(\square\) no  
   law

\(\text{Neutral\-}yes\ +\)

f) \(a + 0 = a\)  
   \[3 + 0 = 3 \quad \ldots \quad 3 = 3\]  
   \(\checkmark\) yes \(\square\) no  
   law

\(\text{Neutral\-}yes\ +\)

g) \(a - a = 0\)  
   \[3-3 = 0 \quad \ldots \quad 0 = 0\]  
   \(\checkmark\) yes \(\square\) no  
   law

2) Which laws (theorems) not yet mentioned above do you remember?
Students know theorems

werden Sie hier nur *grunpiegenaes* wissen gefragt, das Sie an Anrangerunterricht erinnern wird.

1) Hier sind einfache Rechensätze. (i) *Können Sie sich an einige Rechengesetze erinnern?*  
(ii) Wissen Sie ihre Namen noch? (iii) Können Sie die Gesetze auf Zahlen anwenden?

a) \( a + b = b + a \)  
(i) \( X \) ja \( \square \) nein  
(ii) Gesetz der Vertauschung für +

(iii) \( 2 + 3 = 3 + 2 \) \( \ldots \) \( 5 = 5 \)

b) \( a \cdot b = b \cdot a \)  
\( \blacksquare \) ja \( \square \) nein  
Gesetz der Vertauschung für ·

\( 2 \cdot 3 = 3 \cdot 2 \) \( \ldots \) \( 6 = 6 \)

c) \( (a + b) + c = a + (b + c) \)  
\( \blacksquare \) ja \( \square \) nein  
Gesetz ..............................................

\( (1 + 2) + 3 = 1 + (2 + 3) \) \( \ldots \) \( 6 = 6 \) \( \ldots \) \( \ldots = \ldots \)

d) \( a \cdot (b + c) = a \cdot b + a \cdot c \)  
\( \blacksquare \) ja \( \square \) nein  
Gesetz ..............................................

\( 2 \cdot (1 + 3) = 2 \cdot 1 + 2 \cdot 3 \) \( \ldots \) \( 2 \cdot 4 = 2 + 6 \) \( \ldots \) \( 8 = 8 \)

e) \( a \cdot 1 = a \)  
\( \blacksquare \) ja \( \square \) nein  
Gesetz ..............................................

\( 4 \cdot 1 = 4 \) \( \ldots \) \( 4 = 4 \)

f) \( a + 0 = a \)  
\( \blacksquare \) ja \( \square \) nein  
Gesetz ..............................................

\( 2 + 0 = 2 \) \( \ldots \) \( 2 = 2 \)

g) \( a - a = 0 \)  
\( \blacksquare \) ja \( \square \) nein  
Gesetz ..............................................

\( 1 - 1 = 0 \) \( \ldots \) \( 0 = 0 \)

2) Erinnern Sie sich auch an andere Gesetze (die Sie nicht oben angemerkt finden)?
Students know theorems

matemática. Esta formado por preguntas sobre conocimientos básicos.

1) A continuación encontrarás las reglas básicas del álgebra (es decir, teoremas).
(a) ¿Te acuerdas de alguna regla? (b) ¿Te acuerdas del nombre de alguna regla? (c) ¿Sabrías aplicar las reglas en casos concretos?

a) \[ a + b = b + a \]  
   (a) sí □ no  
   (c) \[ 2 + 3 = 3 + 2 \ldots 5 = 5 \]

b) \[ a \cdot b = b \cdot a \]
   (b) sí □ no  
   \[ 3 \cdot 2 = 2 \cdot 3 \ldots 6 = 6 \]

c) \[ (a + b) + c = a + (b + c) \]
   (b) sí □ no  
   \[ (3+2)+5 = 3+(2+5) \ldots 5+5 = 3+7 \ldots 10 = 10 \]

d) \[ a \cdot (b + c) = a \cdot b + a \cdot c \]
   (a) sí □ no  
   \[ 2\cdot(3+4) = 2\cdot3+2\cdot4 \ldots 2\cdot7 = 6+8 \ldots 14 = 14 \]

e) \[ a \cdot 1 = a \]
   (a) sí □ no  
   \[ 5 \cdot 1 = 5 \ldots 5 = 5 \]

f) \[ a + 0 = a \]
   (a) sí □ no  
   \[ 4+0 = 4 \ldots 4 = 4 \]

g) \[ a - a = 0 \]
   (a) sí □ no  
   \[ 1-1 = 0 \ldots 0 = 0 \]
Students know theorems

matematičkog obrazovanja.

1) U nasvaku su navedeni neki od osnovnih algebarskih zakona (teorema). (a) Da li se sećate nekih od ovih zakona? (b) Da li se sećate njihovih imena? (c) Umete li da primenite ove zakone na brojeve?

**Primer:** $a + b = b + a$  
(a) da ne  
(c) $2 + 3 = 3 + 2$  
5 = 5

a) $a \cdot b = b \cdot a$  
da ne  
$
\begin{array}{c}
1 \cdot 3 = 3 \\
15 = 15
\end{array}$

b) $(a + b) + c = a + (b + c)$  
da ne  
$
\begin{array}{c}
5 + 2 + 3 = 5 + (2 + 3) \\
10 = 10
\end{array}$

c) $a \cdot (b + c) = a \cdot b + a \cdot c$  
da ne  
$
\begin{array}{c}
5 \cdot (2 + 3) = 5 \cdot 2 + 5 \cdot 3 \\
25 = 25
\end{array}$

d) $a \cdot 1 = a$  
da ne  
$
\begin{array}{c}
5 \cdot 1 = 5 \\
5 = 5
\end{array}$

e) $a + 0 = a$  
da ne  
$
\begin{array}{c}
5 + 0 = 5 \\
5 = 5
\end{array}$

f) $a - a = 0$  
da ne  
$
\begin{array}{c}
5 - 5 = 0 \\
0 = 0
\end{array}$

2) Možete li navesti još neke zakone koji nisu gore navedni?
Students know theorems

- Front-side: students can simplify
  - They perfectly **operate on formulas**
  - Do they have a notion of theorem?
  - I.e. do they comprehend formal language?

- Front-side: students **know theorems**
  - They demonstrate comprehension of “theorem/rule”:
    - *one rule for all numbers*
  - All great !!!!!?
Students know theorems

- Front-side: students can simplify
  - They perfectly operate on formulas
  - Do they have a notion of theorem?
  - I.e. do they comprehend formal language?
- Front-side: students know theorems
  - They demonstrate comprehension of “theorem/rule”:
    one rule for all numbers
  - All great !!!!?
Students know theorems

- Front-side: students can simplify
  - They perfectly operate on formulas
  - Do they have a notion of theorem?
  - I.e. do they comprehend formal language?

- Front-side: students know theorems
  - They demonstrate comprehension of “theorem/rule”: one rule for all numbers
  - All great !!!!?
Students know theorems

- Front-side: students can simplify
  - They perfectly operate on formulas
  - Do they have a notion of theorem?
  - I.e. do they comprehend formal language?
- Front-side: students know theorems
  - They demonstrate comprehension of “theorem/rule”: one rule for all numbers
  - All great !!!!?
1 Mathematics and language

2 An EU-wide questionnaire
   Organisational framework
   Front-side: students can simplify
   Front-side: students know theorems
   Back-side: students cannot use theorems

3 Consequences...
   ... for didactics research
   ... for R&D on math assistants

4 Summary
Theorem justifying steps in algebraic transformations

University of... (if possible, change icon accordingly)

Mathematics is considered a difficult subject. This questionnaire is part of a research on basic principles and reasons for difficulties in mathematics education. Thus you are just asked basic knowledge, most of which will remind you of early mathematics classes.

1) Below you find basic laws of algebra (i.e. theorems), a) Do you remember some laws? (b) Do you even remember the names of some laws? (c) Can you apply the laws to numbers?
   a) \( a + b = b + a \) (yes \( \square \) no \( \bigcirc \)) law: commutativity for +
   b) \( a \cdot b = b \cdot a \) (yes \( \square \) no \( \bigcirc \)) law:
   c) \( a + (b + c) = a + (b + c) \) (yes \( \square \) no \( \bigcirc \)) law:
   d) \( a \cdot (b + c) = a \cdot b + a \cdot c \) (yes \( \square \) no \( \bigcirc \)) law:
   e) \( a \cdot 0 = 0 \) (yes \( \square \) no \( \bigcirc \)) law:
   f) \( a \cdot 1 = a \) (yes \( \square \) no \( \bigcirc \)) law:
   g) \( a + 0 = a \) (yes \( \square \) no \( \bigcirc \)) law:
   h) \( a - a = 0 \) (yes \( \square \) no \( \bigcirc \)) law:

2) Do you remember any other mathematical laws, not yet mentioned above?
   a) ............................................................. law: .............................................................
   b) ............................................................. law: .............................................................
   c) ............................................................. law: .............................................................

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra, but usually one simplifies without laws, for instance:
   a) \( 2(x + 3) - 6y - 2 x + 6 y - 6y - 6y \)
   b) \( 2(x + 3) + 6y = \)
   c) \( r + r(2 + s) = \)
   d) \( u + 1 \cdot (u - 1) = \)
   e) \( (x + y) \cdot (x - y) = \)

4) Similarly describe a stepwise justification of the following simplifications, please; Take as many steps as needed:
   a) \( 2(x + 3) + 6y = \)
   b) \( r + r(2 + s) = \)
   c) \( u + 1 \cdot (u - 1) = \)

5) Can the simplification \( (x + y)(x - y) = x^2 - y^2 \) be justified using the above laws only?
   a) If "yes", give the first three steps and justifications, please:
   b) If "no", give some missing laws, please:

6) I am a student in a degree course for mathematics (at university) \( \bigcirc \) yes \( \square \) no

Thank you for your attention!

For results see http://www.ist.tugraz.at/projects/issac/www/content/status.html/quest10
This page is about using laws to justify steps in simplifications. We give the following abbreviations for laws:

- **[C+]** \( a + b = b + a \)
- **[A++]** \( (a + b) + c = a + (b + c) \)
- **[A+-]** \( (a + b) - c = a + (b - c) \)
- **[U+]** \( a + 0 = a \)
- **[D+]** \( a \cdot (b + c) = a \cdot b + a \cdot c \)
- **[C-]** \( a \cdot b = b \cdot a \)
- **[A-]** \( (a \cdot b) \cdot c = a \cdot (b \cdot c) \)
- **[I+]** \( a - a = 0 \)
- **[U-]** \( a \cdot 1 = a \)
- **[D-]** \( a \cdot (b - c) = a \cdot b - a \cdot c \)

Here is an example of stepwise justifying a simplification by use of these laws and by calculating natural numbers \([\mathbb{N}^+ \cdot -]\):

\[
2 \cdot (x + 3 \cdot y) - 6 \cdot y = [D+] = (2 \cdot x + 2 \cdot (3 \cdot y)) - 6 \cdot y = [A-] = (2 \cdot x + (2 \cdot 3) \cdot y) - 6 \cdot y = [N+] = (2 \cdot x + 6 \cdot y) - 6 \cdot y = [A+] = 2 \cdot x + (6 \cdot y - 6 \cdot y) = [I+] = 2 \cdot x + 0 = [U+] = 2 \cdot x.
\]

4) **Similarly describe a stepwise justification of the following simplifications, please;**

Take as many steps you need:

a) \( 2 \cdot (x + 3 \cdot y) + 6 \cdot y = \)

b) \( r + r \cdot (2 + s) = \)
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Neuper

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Theorems justifying steps in algebraic transformations
University of ................................ (if possible, change icon accordingly)

Mathematics is considered a difficult subject. This questionnaire is part of a research on basic principles and reasons for difficulties in mathematics education. Thus you are just asked basic knowledge, most of which will remind you of early mathematics classes.

1) Below you find basic laws of algebra (i.e. theorems). (a) Do you remember some laws? (b) Do you even remember the names of some laws? (c) Can you apply the laws to numbers?

- a + b = b + a (X yes ☐ no) law of commutativity for +
- a + b + c = a + (b + c) ☐ yes ☐ no law
- a + b = a + b (☐ yes ☐ no) law
- a + b + c = a + (b + c) ☐ yes ☐ no law
- a + b + c = a + (b + c) ☐ yes ☐ no law
- a + 0 = a ☐ yes ☐ no law
- a - 0 = a ☐ yes ☐ no law

2) Do you remember any other mathematical laws, not yet mentioned above?
- ........................................................ law
- ........................................................ law
- ........................................................ law

3) Simplify the following algebraic expressions, please. Simplifying such expressions is learned together with laws of algebra, but usually one simplifies without laws, for instance:
- 2(x + 3) - 6y = 2x + 6 - 6y
- 2(x + 3) + 6y = ........................................................
- 2(x + 3) + 6y = ........................................................
- 2(x + 3) + 6y = ........................................................
- 2(x + 3) + 6y = ........................................................

4) Similarly describe a stepwise justification of the following simplifications, please; Take as many steps as you need:
- 2(x + 3) - 6y = ........................................................
- 2(x + 3) - 6y = ........................................................
- 2(x + 3) - 6y = ........................................................
- 2(x + 3) - 6y = ........................................................

5) Can the simplification (x+y)(x-y) be justified using the above laws only?
- If "yes", give the first three steps and justifications, please:
  (x+y)(x-y) ...........................................................
  ...........................................................
  ...........................................................
- If "no", give some missing laws, please:
  ...........................................................
  ...........................................................
  ...........................................................

6) I am a student in a degree course for mathematics (at university) ☐ yes ☐ no

Thank you for your attention!
For results see http://www.ist.tugraz.at/projects/issac/www/content/status.html#quest10
Cannot use theorems

5) Can the simplification \((x+y)(x-y) = x \cdot x - y \cdot y\) be justified using the above laws only?
   a) If "yes", give the first three steps and justifications, please:
      \[(x+y)(x-y) = \] 
      \[ = \] 
      \[ = \]

   b) If "no", give some missing laws, please:
      \[\text{law} \] 
      \[\text{law} \] 
      \[\text{law} \] 
      \[\text{law} \]

6) I am a student in a degree course for mathematics (at university) □ yes □ no

Thank you for your attention!
4) Similarly describe a stepwise justification of the following simplifications, please:
Take as many steps you need:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = \quad [D^+] = \quad \text{Simplify} \quad (2 \cdot x + 2 \cdot (3 \cdot y)) + 6y \]
\[ = [A^+] = (2 \cdot x + (2 \cdot 3 \cdot y)) + 6y \]
\[ = [N^+] = (2 \cdot x + 6 \cdot y) + 6 \cdot y = [N^+] = 2 \cdot x + 6y + 6y = [N^+] = 2 \cdot x + 12y \]

b) \[ r + r \cdot (2 + s) = \quad [D^+] = \quad \text{Simplify} \quad r + r \cdot 2 + r \cdot s \]
\[ = [A^+] = r + (2 \cdot r + s) \]
\[ = [N^+] = r + 3 \cdot r + s = [N^+] = 3 \cdot r + s \cdot r \]
\[ = [D^+] = (3 + s) \cdot r \]

c) \[ (u + 1) \cdot (u - 1) = \quad [C^+] = \quad \text{Simplify} \quad (u + 1) \cdot u - (u + 1) \cdot 1 \]
\[ = [C^+] = u \cdot (u + 1) - (u + 1) \cdot 1 \]
\[ = [N^+] = u \cdot (u + 1) - (u + 1) \cdot 1 = [N^+] = u \cdot u + u - u - 1 \]

5) Can the simplification \((x+y) \cdot (x-y) = x \cdot x - y \cdot y\) be justified using the above laws only?

a) If “yes”, give the first three steps and justifications, please:
\[ (x+y)(x-y) = [D^+] = \quad \text{Simplify} \quad (x+y) \cdot x - (x+y) \cdot y \]
\[ = [C^+] = x \cdot (x+y) - y \cdot (x+y) \]
\[ = [A^+] = x \cdot x + x \cdot y - (y \cdot x + y \cdot y) = [C^+] = x \cdot x + (x \cdot y - y \cdot x) + y \cdot y \]

b) If “no”, give some missing laws, please:
Begründen Sie bitte ebenso beim schrittweisen Vereinfachen; machen Sie so viele Schritte wie Sie brauchen:

a) \( 2 \cdot (x + 3 \cdot y) + 6 \cdot y = (2 \cdot x + 2 \cdot 3 \cdot y) + 6 \cdot y = (2 \cdot x + 6 \cdot y) + 6 \cdot y = 2 \cdot x + 12 \cdot y \)

b) \( r + r \cdot (2 + s) = r + (2r + rs) = 3r + rs \)

c) \( (u + 1) \cdot (u - 1) = u^2 - 1 \)

5) Lässt sich \( (x+y) \cdot (x-y) = x \cdot x - y \cdot y \) nur mit obigen Gesetzen begründen?

a) Wenn "ja", geben Sie bitte die ersten drei Schritte samt Begründung an:

\( (x+y)(x-y) = xx + yx - yx - yy \)

b) Wenn "nein", geben Sie bitte einige der fehlenden Gesetze an:
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\[\begin{align*}
[N] &= (2 \cdot x + 6 \cdot y) - 6 \cdot y \\
[A+] &= 2 \cdot x + (6 \cdot y - 6 \cdot y) \\
[I+] &= 2 \cdot x + 0 \\
[U+] &= 2 \cdot x
\end{align*}\]

4) Begründen Sie bitte ebenso beim schrittweisen Vereinfachen; machen Sie so viele Schritte wie Sie brauchen:

a) \[2 \cdot (x + 3 \cdot y) + 6 \cdot y = \]
\[\begin{align*}
[A^+] &= 2 \cdot x + 6 \cdot y \\
[A^-] &= 2 \cdot x + 12 \cdot y
\end{align*}\]

b) \[r + r \cdot (2 + s) = \]
\[\begin{align*}
[A^+] &= 3 \cdot r + r \cdot s \\
[D^-] &= r \cdot (s + s)
\end{align*}\]

c) \[(u + 1) \cdot (u - 1) = \]
\[\begin{align*}
[D^-] &= u^2 - u + 1 \cdot (u - 1) \\
[L^+] &= u^2 - 1
\end{align*}\]

5) Lässt sich \((x+y) \cdot (x-y) = x \cdot x - y \cdot y\) nur mit obigen Gesetzen begründen?

a) Wenn “ja”, geben Sie bitte die ersten drei Schritte samt Begründung an:
\[\begin{align*}
(x+y)(x-y) &= \]
\[\begin{align*}
[D^-] &= x^2 - x \cdot y + y \cdot (x-y) \\
[I^-] &= x^2 - x \cdot y + y \cdot y
\end{align*}\]

b) Wenn “nein”, geben Sie bitte einige der fehlenden Gesetze an:
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**Cannot use theorems**

\[
\begin{align*}
(2 \cdot x + 6 \cdot y) - 6 \cdot y &= 2 \cdot x + (6 \cdot y - 6 \cdot y) \\
&= 2 \cdot x + 0 = 2 \cdot x
\end{align*}
\]

4) Begründen Sie bitte ebenso beim schrittweisen Vereinfachen; machen Sie so viele Schritte wie Sie brauchen:

a) \[2 \cdot (x + 3 \cdot y) + 6 \cdot y = \left[ A^+ \right] = 2x + 6y + 6y = 2x + 12y\]
   \[\left[ D^+ \right] = 2x + 12y\]
   \[\left[ \ldots \right] = 2x + 12y\]
   \[\left[ \ldots \right] = 2x + 12y\]

b) \[r + r \cdot (2 + s) = \left[ A^+ \right] = r + 2r + 2s\]
   \[\left[ A^{++} \right] = r + 2s + 2s\]
   \[\left[ D^+ \right] = 2s + 2s\]
   \[\left[ \ldots \right] = 3s + 2s\]

5) Lässt sich \((x+y) \cdot (x-y) = x \cdot x - y \cdot y\) nur mit obigen Gesetzen begründen?

a) Wenn "ja", geben Sie bitte die ersten drei Schritte samt Begründung an:
   \[
x^2 - x \cdot y = \left[ D^- \right] = x^2 + xy - xy - y^2
   = \left[ A^+ \right] = x^2 - y^2 = x \cdot x - y \cdot y
   \]

b) Wenn "nein", geben Sie bitte einige der fehlenden Gesetze an:
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**Cannot use theorems**

\[
2 \cdot (x + 3 \cdot y) - 6 \cdot y \not\equiv (2 \cdot x + 2 \cdot (3 \cdot y)) - 6 \cdot y \not\equiv (2 \cdot x + (2 \cdot 3) \cdot y) - 6 \cdot y \\
= [N] = (2 \cdot x + 6 \cdot y) - 6 \cdot y = [A+] = 2 \cdot x + (6 \cdot y - 6 \cdot y) = [l+] = 2 \cdot x + 0 = [U+] = 2 \cdot x
\]

4) Tomando la simplificación anterior como ejemplo, simplifica las siguientes expresiones algebraicas indicando en cada paso la regla usada.
Usa tantos pasos como consideres necesarios:

a) \(2 \cdot (x + 3 \cdot y) + 6 \cdot y\)
   
   \[
   = [D+] = 2 \cdot x + 6 \cdot y + 6 \cdot y
   
   = [..] = 2 \cdot x + 12 \cdot y
   
   = [..] =
   
   \]

b) \(r \cdot r \cdot (2 + s)\)
   
   \[
   = [D+] = r \cdot 2 \cdot r \cdot r \cdot s
   
   = [D+] = 3 \cdot r + r \cdot s
   
   = [D+] = (3 + s) \cdot r
   
   = [..] =
   
   \]

c) \((u + 1) \cdot (u - 1)\)
   
   \[
   = [..] = u^2 - 1
   
   = [..] =
   
   \]

5) ¿Puede la transformación \((x + y) \cdot (x - y) = x \cdot x - y \cdot y\) justificarse usando únicamente las reglas mencionadas más arriba?

a) Si tu respuesta es “sí”, escribe los tres primeros pasos indicando la regla usada en cada paso:
   \[
   (x+y)(x-y) = [D+] = (x \cdot x - x \cdot y) + (y \cdot x - y \cdot y) = [T+] =
   
   = [..] = x^2 - y^2
   
   = [..] =
   
   \]
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\[2(x + 3y) - 6y = (2x - 6y) = (2x + 6y - 6y) = 2x + 0 = 2x\]

4) Tomando la simplificación anterior como ejemplo, simplifica las siguientes expresiones algebraicas indicando en cada paso la regla usada.
Usa tantos pasos como consideres necesarios:

a) \[2(x + 3y) + 6y = (2x + 2(3y)) + 6y\]
   \[= (2x + 2(3y)) + 6y = (2x + 6y) + 6y = 2x + 12y\]

b) \[r + r(2 + s) = (r + 2r + r\cdot s)\]
   \[= r + 2r + r\cdot s = r(2 + s)\]

c) \[(u + 1)(u - 1) = (u\cdot u - u\cdot 1) + (u\cdot 1 - 1\cdot 1)\]
   \[= (u^2 - u) + (u - 1) = (u^2 - u) + (u - u) = u^2 - 1\]

5) ¿Puede la transformación \((x + y) \cdot (x-y) = x\cdot x - y\cdot y\) justificarse usando únicamente las reglas mencionadas más arriba?
   a) Si tu respuesta es “sí”, elige los tres primeros pasos indicando la regla usada en cada paso:
      \[(x+y)(x-y) = [(x \cdot x) - (x \cdot y)] + [(y \cdot x) - (y \cdot y)]\]
      \[= (x^2 - xy) + (xy - y^2) = x^2 - y^2\]
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\[
\begin{align*}
2(x + 3y) - 6y &= \left(2x + 2(3y)\right) - 6y \\
&= 2x + 6y - 6y \\
&= 2x \\
\end{align*}
\]

4) Tomando la simplificación anterior como ejemplo, simplifica las siguientes expresiones algebraicas indicando en casa paso la regla usada.
Usa tantos pasos como consideres necesarios:

a) \[2(x + 3y) + 6y = \]

\[
\begin{align*}
&= [A_+] \\
&= 2x + 6y + 6y \\
&= [A_+] \\
&= 2x + 12y \\
&= [D_+] \\
&= 2x + 12y \\
\end{align*}
\]

b) \[r + r(2 + s) = \]

\[
\begin{align*}
&= [A_+] \\
&= r + (2r + sr) \\
&= [A_+] \\
&= 3r + sr \\
&= [D_+] \\
&= (3 + s) \cdot r \\
\end{align*}
\]

c) \[(u + 1)(u - 1) = \]

\[
\begin{align*}
&= [I_+] \\
&= u^2 - 1 \\
&= [I_+] \\
&= u^2 - 1 \\
\end{align*}
\]

5) ¿Puede la transformación \((x + y) \cdot (x-y) = x \cdot x - y \cdot y\) justificarse usando únicamente las reglas mencionadas más arriba?

a) Si tu respuesta es “sí”, escribe los tres primeros pasos indicando la regla usada en cada paso:

\[
(x+y)(x-y) = [D_+] = x^2 - x \cdot y + x \cdot y - y^2 \\
&= [I_+] = x^2 - y^2 \\
\]


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Summary

4) Kao u navedenom primeru, uprости izraze u navođenje zakona:
Uradite koliko god koraka je potrebno:

a) \[2 \cdot (x + 3 \cdot y) + 6 \cdot y = \begin{aligned} &\cdot (2 \cdot x + 2 \cdot (3 \cdot y)) + 6 \cdot y \\
&\cdot = \cdot (2 \cdot x + 6 \cdot y) - 6 \cdot y \\
&\cdot =^\text{[N]} = (2 \cdot x + 6 \cdot y) + 6 \cdot y \\
&\cdot =^\text{[A+]} = 2 \cdot x + (6 \cdot y + 6 \cdot y) \\
&\cdot =^\text{[I+]} = 2 \cdot x + 12 \cdot y \\
&\cdot =^\text{[U+]} = 2 \cdot x + 12 \cdot y
\]

b) \[r + r \cdot (2 + s) = \begin{aligned} &\cdot (r + 2 \cdot r) + r \cdot s \\
&\cdot = \cdot (r + 2 \cdot r) + r \cdot s \\
&\cdot =^\text{[N]} = 3 \cdot r + r \cdot s \\
&\cdot =^\text{[A+]} = 3 r + 3 r \cdot s
\]

c) \[(u + 1) \cdot (u - 1) = \begin{aligned} &\cdot (u \cdot (u-1) + 1 \cdot (u-1)) \\
&\cdot = \cdot (u \cdot (u-1) + 1 \cdot (u-1)) \\
&\cdot =^\text{[N]} = (u \cdot u - u + 1 \cdot u - 1) \\
&\cdot =^\text{[A+]} = (u \cdot u - u + 1 \cdot u - 1) - 1 \cdot u + 0 - 1 \cdot 1 \\
&\cdot =^\text{[I+]} = u \cdot u - 1
\]

5) Da li se upršćavanje \((x+y) \cdot (x-y) = x \cdot x - y \cdot y\) može opravdati samo navedenim zakonima?

a) Ako mislite da može, navedite prva 3 koraka:
\[(x+y)(x-y) = \begin{aligned} &\cdot (x \cdot (x-y) + y \cdot (x-y)) \\
&\cdot = \cdot (x \cdot x - y \cdot x + y \cdot x - y \cdot y) \\
&\cdot =^\text{[N]} = (x \cdot x - y \cdot x + y \cdot x - y \cdot y) \\
&\cdot =^\text{[A+]} = x \cdot x - y \cdot x + y \cdot x - y \cdot y \\
&\cdot =^\text{[I+]} = x \cdot x - y \cdot y
\]
Cannot use theorems

\[ 2 \cdot (x + 3 \cdot y) - 6 \cdot y = (2 \cdot x + 2 \cdot (3 \cdot y)) - 6 \cdot y = (2 \cdot x + 2 \cdot 3 \cdot y) - 6 \cdot y \]
\[ = [N] = (2 \cdot x + 6 \cdot y) - 6 \cdot y = [A+] = 2 \cdot x + (6 \cdot y - 6 \cdot y) = [I+] = 2 \cdot x + 0 = [U+] = 2 \cdot x \]

4) Kao u navedenom primeru, uprostite izraze u navođenje zakona:
Uradite koliko god koraka je potrebno:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y \]
\[ = \[D+\] = (2 \cdot x + 2 \cdot (3 \cdot y)) + 6 \cdot y \]
\[ = \[H-\] = (2 \cdot x + 6 \cdot y) + 6 \cdot y \]
\[ = \[A+\] = 2 \cdot x + (6 \cdot y + 6 \cdot y) = 2 \cdot x + 12 \cdot y \]

b) \[ r + r \cdot (2 + s) \]
\[ = \[D+\] = r + (2 \cdot r + r \cdot s) \]
\[ = \[A+\] = r + 2 \cdot r + r \cdot s \]
\[ = \[D-\] = r \left( 1 + 2 + s \right) \]

5) Da li se uprošćavanje \((x+y) \cdot (x-y) = x \cdot x - y \cdot y\) može opravdati samo navedenim zakonima?

a) Ako mislite da može, navedite prva 3 koraka:
\[ (x+y)(x-y) \]
\[ = \[D+\] = x \cdot x - x \cdot y + y \cdot x - y \cdot y \]
\[ = \[A-\] = x \cdot x - y \cdot y \]

...
Cannot use theorems

\[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = 2 \cdot x + 2 \cdot 3 \cdot y + 6 \cdot y \]
\[ = (2 \cdot x + 6 \cdot y) - 6 \cdot y = 2 \cdot x + (6 \cdot y - 6 \cdot y) = 2 \cdot x + 0 = 2 \cdot x \]

4) Kao u navedenom primeru, uprostite izraze u navođenje zakona:
Uradite koliko god koraka je potrebno:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y \]
\[ = (2 \cdot x + 2 \cdot 3 \cdot y) + 6 \cdot y \]
\[ = (2 \cdot x + 6 \cdot y) + 6 \cdot y \]
\[ = 2 \cdot x + 12 \cdot y \]

b) \[ r + r \cdot (2 + s) \]
\[ = \]
\[ = \]
\[ = \]
\[ = \]

5) Da li se uprošćavanje \((x+y)\cdot(x−y) = x\cdot x − y\cdot y\) može opravdati samo navedenim zakonima?

a) Ako mislite da može, navedite prva 3 koraka:
\[ (x+y)(x−y) = x^2 − y\cdot x + y\cdot x − y^2 \]
\[ = x^2 − y^2 \]
Irreplacable Role of Math. Assistants
Walther Neuper

Mathematics – language

Querstionnaire
Framework
Simplify
Know theorems
Use theorems

Consequences
Psychology
Techn. R&D

Summpary

Cannot use theorems

\[ 2 \cdot (x + 3 \cdot y) - 6 \cdot y + (2 \cdot x + 2 \cdot (3 \cdot y)) - 6 \cdot y = \[N\] = (2 \cdot x + 6 \cdot y) - 6 \cdot y = \[A^+\] = 2 \cdot x + (6 \cdot y - 6 \cdot y) = \[H\] = 2 \cdot x + 0 = \[U^+\] = 2 \cdot x \]

4) Similarly describe a stepwise justification of the following simplifications:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = \[D^+\] = 2 \cdot x + 2 \cdot 3 \cdot y + 6 \cdot y = \[N^*\] = 2 \cdot x + 6 \cdot y + 6 \cdot y \]

b) \[ r + r \cdot (2 + s) = \[D^+\] = r + (2 \cdot r + r \cdot s) = \[N^+\] = 3 \cdot r + r \cdot s \]

5) Can the simplification \((x + y) \cdot (x - z) = x \cdot x - y \cdot y\) be justified using the above laws only?

a) If “yes”, give the first three steps and justifications:

\[ (x + y)(x - y) = \[A^+\] = x \cdot x - x \cdot y + y \cdot x - y \cdot y = \[C^+\] = x \cdot x - x \cdot y + y \cdot y - y \cdot y \]

b) If “no”, give some missing laws:
Cannot use theorems

\[ 2 \cdot (x + 3 \cdot y) - 6 \cdot y = \] 
\[ = \frac{2}{(x + (2 \cdot 3 \cdot y)) - 6 \cdot y} = \frac{2}{x + (2 \cdot 3 \cdot y) - 6 \cdot y} = \frac{2}{x + 0} = \frac{2}{x} \]

4) Similarly describe a stepwise justification of the following simplifications:

a) \( 2 \cdot (x + 3 \cdot y) + 6 \cdot y = \)
\[ = \frac{2 \cdot (x + 3 \cdot y) + 6 \cdot y}{x + 3 \cdot y + 6 \cdot y} = \frac{2 \cdot x + 3 \cdot x \cdot y + 6 \cdot y}{x + 3 \cdot y + 6 \cdot y} \]

b) \( r + r \cdot (2 + s) = \)
\[ = \frac{r + (2r + rs)}{2r + rs + r} = \frac{r + 2r + rs}{2r + rs + r} \]

c) \( (u + 1) \cdot (u - 1) = \)
\[ = \frac{u^2 - 1u + 1u - 1}{u^2 - 1} = \frac{u^2 - 1}{u^2 - 1} \]

5) Can the simplification \( (x+y) \cdot (x-z) = x \cdot x - y \cdot y \) be justified using the above laws only?

a) If “yes”, give the first three steps and justifications:
\[ (x+y)(x-z) = \]
\[ = \frac{(x+y)(x-z)}{x^2 - y^2} = \frac{x^2 - y^2}{x^2 - y^2} = x \cdot x - y \cdot y \]

b) If “no”, give some missing laws:
4) Similarly describe a stepwise justification of the following simplifications:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = \]
\[ = \]
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Cannot use theorems

4) Kao u navedenom primeru, uprostite izraze u navođenje zakona:
Uradite koliko god koraka je potrebno:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y \]
   \[= [A+] = (2 \cdot x + 2(3 \cdot y)) + 6 \cdot y \]
   \[= [A+] = 2 \cdot x + (2 \cdot 3 \cdot y) + 6 \cdot y \]
   \[= [A+] = 2 \cdot x + (6 \cdot y + 6 \cdot y) \]
   \[= [U+] = 2 \cdot x + 0 \]
   \[= [U+] = 2 \cdot x \]

b) \[ r + r \cdot (2 + s) \]
   \[= [A+] = r + (2 \cdot r + rs) \]
   \[= [A+] = (r + 2 \cdot r) + rs \]
   \[= [A+] = 3 \cdot r + rs \]
   \[= [A+] = 3 \cdot r + rs \]
   \[= [A+] = r \cdot (3 + s) \]

5) Da li se uprošćavanje \((x+y) \cdot (x-y) = x \cdot x - y \cdot y\) može opravdati samo navedenim zakonima?

a) Ako mislite da može, navedite prva 3 koraka:
\[ (x+y)(x-y) \]
\[= [A+] = (x+y) \cdot x - (x+y) \cdot y \]
\[= [A+] = x \cdot x + y \cdot x - x \cdot y - y \cdot y \]
\[= [A+] = x^2 + xy - xy - y^2 \]
\[= [A+] = x^2 - y^2 \]
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Cannot use theorems

\[ 2 \cdot (x + 3 \cdot y) - 6 \cdot y = (2 \cdot x + 2 \cdot (3 \cdot y)) - 6 \cdot y = (2 \cdot x + (2 \cdot 3) \cdot y) - 6 \cdot y = (2 \cdot x + 6 \cdot y) - 6 \cdot y = (2 \cdot x + 6 \cdot y - 6 \cdot y) = (2 \cdot x + 0) = 2 \cdot x \]

4) Tomando la simplificación anterior como ejemplo, simplifica las siguientes expresiones algebraicas indicando en cada paso la regla usada. Usa tantos pasos como consideres necesarios:

a) \( 2 \cdot (x + 3 \cdot y) + 6 \cdot y = \)
\[ = [A+]+ = 7 \cdot x + (6 \cdot 3 \cdot y) + 6 \cdot y = [A+]+ = 7 \cdot x + 17 \cdot y \]

b) \( r + r \cdot (2 + s) = \)
\[ = [A+]+ = (r + 2 \cdot r) + r \cdot s = [A+]+ = 3 \cdot r + r \cdot s \]

5) ¿Puede la transformación \((x + y) \cdot (x-y) = x^2 - y^2\) justificarse usando únicamente las reglas mencionadas más arriba?

a) Si tu respuesta es “sí”, escribe los tres primeros pasos indicando la regla usada en cada paso:
\[ (x+y)(x-y) = (x+y) \cdot (x-y) = \]
\[ = [D+] = x^2 - y^2 \]
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\[ 2 \cdot (x + 3 \cdot y) - 0 \cdot y = \underbrace{(2 \cdot x + 2 \cdot (3 \cdot y)) - 0 \cdot y =}_{[N]} (2 \cdot x + (2 \cdot 3) \cdot y) - 0 \cdot y = [I^+] = 2 \cdot x + 0 = [U^+] = 2 \cdot x. \]

4) **Begründen Sie bitte ebenso beim schrittweisen Vereinfachen;** machen Sie soviele Schritte wie Sie brauchen:

a) \[ 2 \cdot (x + 3 \cdot y) + 6 \cdot y = [D^+] = (2 \cdot x + 2 \cdot (3 \cdot y) + 6 \cdot y \]
\[ = [A^-] = (2 \cdot x + (2 \cdot 3) \cdot y) + 6 \cdot y \]
\[ = [N^-] = 2 \cdot x + 6 \cdot y + 6 \cdot y \]
\[ = [N^+] = 2 \cdot x + 12 \cdot y. \]

b) \[ r + r \cdot (2 + s) = [D^+] = r + 2r + r \cdot s \]
\[ = [\ldots] = r + 2r + r \cdot s \]
\[ = [\ldots] = 3r + r \cdot s \]
\[ = [N^+] = 3r + r \cdot s. \]

c) \[ (u + 1) \cdot (u - 1) = [D^-] = (u + 1) \cdot u - (u + 1) \cdot 1 \]
\[ = [D^+] = u^2 + u - u - 1 \cdot 1 \]
\[ = [\ldots] = u^2 - 1 \]
\[ = [N^+] = u^2 - 1. \]

5) **Lässt sich** \((x+y) \cdot (x-y) = x \cdot x - y \cdot y \) **nur mit obigen Gesetzen begründen?**

a) Wenn „ja“, geben Sie bitte die ersten drei Schritte samt Begründung an:
\[ (x+y)(x-y) = [D^-] = (x+y) \cdot x - (x+y) \cdot y \]
\[ = [D^+] = x \cdot x + y \cdot x - x \cdot y - y \cdot y \]
\[ = [\ldots] = x \cdot x - y \cdot y \]

b) Wenn „nein“, geben Sie bitte einige der fehlenden Gesetze an:
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Mathematics – language

Questionnaire Framework Simplify Know theorems Use theorems

Consequences Psychology Techn. R&D

Summary

Cannot use theorems

\[ (x + y)(x - y) = x \cdot x - y \cdot y \]

4) Similarly describe a stepwise justification of the following simplifications:
   a) \( 2 \cdot (x + 3 \cdot y) + 6 \cdot y \)
      \( =^{[P+]} = 2 \cdot x + 2 \cdot (3 \cdot y) + 6 \cdot y \)
      \( =^{[A]} = 2 \cdot x + (2 \cdot 3 \cdot y + 6 \cdot y) \)
      \( =^{[U]} = 2 \cdot x + 12 \cdot y \)
   b) \( r \cdot r \cdot (2 + s) \)
      \( =^{[D]} = r \cdot r \cdot (2 + s) \)
      \( =^{[A]} = r \cdot (1 + (2 + s)) \)
      \( =^{[U]} = r \cdot (3 + s) \)
   c) \( (u + 1) \cdot (u - 1) \)
      \( =^{[D]} = (u + 1) \cdot u - (u + 1) \cdot 1 \)
      \( =^{[A]} = u^2 + u - u - 1 \)
      \( =^{[U]} = u^2 - 1 \)

5) Can the simplification \( (x+y) \cdot (x-y) = x \cdot x - y \cdot y \) be justified using the above laws only?
   a) If “yes”, give the first three steps and justifications:
      \( (x+y)(x-y) \)
      \( =^{[D]} = (x+y) \cdot x - (x+y) \cdot y \)
      \( =^{[U]} = x^2 + y^2 - x \cdot y - y \cdot y \)
   b) If “no”, give some missing laws:
      \( x^2 + (x \cdot y - y \cdot y) + y^2 = x^2 + 2 \cdot y \cdot y + y^2 \)
Irreplacable Role of Math. Assistants
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Summpary

Cannot use theorems

• Front-side: students can simplify
  • They perfectly operate on formulas
  • Do they have a notion of theorem?
  • I.e. do they comprehend formal language?

• Front-side: students know theorems
  • They demonstrate comprehension of “theorem/rule”:
    one rule for all numbers
  • All great !!!!!?

• Back-side: students cannot use theorems
  • $2 \cdot a + 3 \cdot a = \cdots = [D^+] = (2 + 3) \cdot a$ < 10%
  • $(x + y) \cdot (x - y) = [D^-] = (x + y) \cdot x - (x + y) \cdot y$ < 20%
Cannot use theorems

- **Front-side: students can simplify**
  - They perfectly operate on formulas
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- **Back-side: students cannot use theorems**
  - \[2 \cdot a + 3 \cdot a = \ldots = [D^+] = (2 + 3) \cdot a < 10\%\]
  - \[(x + y) \cdot (x - y) = [D^-] = (x + y) \cdot x - (x + y) \cdot y < 20\%\]
**Cannot use theorems**

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  - $2 \cdot a + 3 \cdot a = \cdots = [D+] = (2 + 3) \cdot a < 10\%$
  - $(x + y) \cdot (x - y) = [D-] = (x + y) \cdot x - (x + y) \cdot y < 20\%$
Cannot use theorems

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Questionnaire summary

- Front-side: students can simplify
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  - $2 \cdot a + 3 \cdot a = \cdots = [D^+] = (2 + 3) \cdot a < 10\%$
  - $(x + y) \cdot (x - y) = [D^-] = (x + y) \cdot x - (x + y) \cdot y < 20\%$
  - Students eloquently “speak mother tongue” in math, they do not comprehend math as a formal language.
Questionnaire summary

- **Front-side: students can simplify**
  - They perfectly operate on *formulas*
  - Do they have a notion of theorem?
  - I.e. do they comprehend formal language?

- **Front-side: students know theorems**
  - They demonstrate comprehension of “theorem/rule”:
    - *one rule for all* numbers
  - All great !!!!?

- **Back-side: students cannot use theorems**
  - $2 \cdot a + 3 \cdot a = \cdots = [D^+] = (2 + 3) \cdot a < 10\%$
  - $(x + y) \cdot (x - y) = [D^-] = (x + y) \cdot x - (x + y) \cdot y < 20\%$

- Students eloquently “speak mother tongue” in math, they do **not** comprehend math as a formal language.
Questionnaire summary

- Front-side: students can simplify
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  - I.e. do they comprehend formal language?
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  - They demonstrate comprehension of “theorem/rule”: one rule for all numbers
  - All great !!!!?
- Back-side: students cannot use theorems
  - \(2 \cdot a + 3 \cdot a = \cdots =_{[D^+]} (2 + 3) \cdot a < 10\%\)
  - \((x + y) \cdot (x - y) =_{[D^-]} (x + y) \cdot x - (x + y) \cdot y < 20\%\)
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Questionnaire summary

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  - They demonstrate comprehension of “theorem/rule”:
    one rule for all numbers
  - All great !!!!?
- Back-side: students cannot use theorems
  - \(2 \cdot a + 3 \cdot a = \cdots = ^{[D^+]} = (2 + 3) \cdot a\) \(< 10\%\)
  - \((x + y) \cdot (x - y) = ^{[D^-]} = (x + y) \cdot x - (x + y) \cdot y\) \(< 20\%\)
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Questionnaire summary

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- **Back-side: students cannot use theorems**
  - $2 \cdot a + 3 \cdot a = \cdots = [D^+] = (2 + 3) \cdot a \quad < 10\%$
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Questionnaire summary

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  - $2 \cdot a + 3 \cdot a = \ldots = [D^+] = (2 + 3) \cdot a < 10\%$
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Outline

1. Mathematics and language

2. An EU-wide questionnaire
   Organisational framework
   Front-side: students can simplify
   Front-side: students know theorems
   Back-side: students cannot use theorems

3. Consequences . . .
   . . . for didactics research
   . . . for R&D on math assistants

4. Summary
Students do **not** comprehend math as a formal language –

- Let’s better understand **developmental psychology**!
  - Advocate develop. psychology of **language learning**!
  - Understand **mental maturity** for reaching levels: variable - equation - function ... theorem - proof - axiom
  - How **promote** “mother tongue” to formal language?
Didactics research

Students do **not** comprehend math as a formal language –

- Let’s better understand **developmental psychology**!
- Advocate devel.psychology of **language learning**!
- Understand **mental maturity** for reaching levels: variable - equation - function . . . theorem - proof - axiom
- How **promote** “mother tongue” to formal language?

-
Didactics research

Students do **not** comprehend math as a formal language –

- Let’s better understand **developmental psychology**!
- Advocate deve.psycho. of **language learning**!
- Understand **mental maturity** for reaching levels: `variable - equation - function ... theorem - proof - axiom`
- How promote “mother tongue” to formal language?
Didactics research

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Outline

1. Mathematics and language
2. An EU-wide questionnaire
   Organisational framework
   Front-side: students can simplify
   Front-side: students know theorems
   Back-side: students cannot use theorems
3. Consequences . . .
   . . . for didactics research
   . . . for R&D on math assistants
4. Summary
R&D on math assistants

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- Make math assistants “transparent models of math”
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  -
Dialog mechanically generated by CTP-technology

A calculation proceeded to a certain step (no. 3.):

1. \[ \frac{d}{dx}(x^2 + \sin(3 \cdot x^4)) \]

2. \[ 2 \cdot x^{2-1} + \frac{d}{dx} \sin(3 \cdot x^4) \]

3. \[ 2 \cdot x + \frac{d}{dx} \sin(3 \cdot x^4) \]

How can we guide the student to the next formula (no. 4.)

\[ 4. \quad 2 \cdot x + \cos(3 \cdot x^4) \cdot \frac{d}{dx}(3 \cdot x^4) \]

...or some algebraically equivalent formula?
Dialog mechanically generated by CTP-technology

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4. _______________
Dialog mechanically generated by CTP-technology

3. \(2 \cdot x + \frac{d}{dx} \sin(3 \cdot x^4)\)

4. \(2\)
Dialog mechanically generated by CTP-technology

\[
3. \quad 2 \cdot x + \frac{d}{dx} \sin(3 \cdot x^4)
\]

\[
4. \quad 2 \cdot \underline{\phantom{3 \cdot x^4}}
\]
Dialog mechanically generated by CTP-technology

3. \[ 2 \cdot x + \frac{d}{dx} \sin(3 \cdot x^4) \]

4. \[ 2 \cdot x \underline{\phantom{10}} \]
Dialog mechanically generated by CTP-technology

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3. \( 2 \cdot x + \frac{d}{dx} \sin(3 \cdot x^4) \)

4. \( 2 \cdot x + \cos(3 \cdot x^4) \)

Input checked by a prover.
Dialog mechanically generated by CTP-technology

3. \[ 2 \cdot x + \frac{d}{dx} \sin(3 \cdot x^4) \]

\[ \frac{d}{dx} \sin(u) = \cos(u) \cdot \frac{d}{dx} \]

Provers identify and suggest theorems.
Dialog mechanically generated by CTP-technology

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\[ 3. \quad 2 \cdot x + \frac{d}{dx} \sin(3 \cdot x^4) \]

\[
\frac{d}{dx} \cos(x) = -\sin(x) \\
\frac{d}{dx} \sin(u) = \cos(u) \cdot \frac{d}{dx} u \\
\frac{d}{dx} x^n = n \cdot x^{n-1}
\]

Provers operate on theories comprising theorems.
Dialog mechanically generated by CTP-technology

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Provers use “matching” for fill-in gaps.
Dialog mechanically generated by CTP-technology

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Provers check, if a formula can be derived in a context.
R&D on math assistants

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Summary

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We better bridge the gap between school and university! Math assistants are indispensable for bridging the gap.
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Common grounds for modelling mathematics in educational software.
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