Educational Tools as Interactive Models of Mathematics
Towards a Domain Model for Mathematics Assistants

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Workshop on Formal Theorem Proving and Applications, Belgrade Jan.09
1. Theorem Proving — Problem Solving
   - Descriptions and Differences
   - Problem Solving: Examples & Demonstrations
   - Mutuality I: Calculations – (geometric) Constructions

2. Interactions in Problem Solving
   - Three Kinds of Steps
   - Steps in Calculations — Demonstration
   - Mutuality II: Steps in Constructions ?
Outline

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(i) Theorem proving constructs a theorem:

\[ \text{prove} : \text{Logic} \times \text{Context} \times \text{Predicate} \rightarrow \text{Theory} \times \text{Theorem} \]

(ii) Problem solving creates a result (usually not type bool):

\[ \text{solve} : \text{Logic} \times \text{Context} \times \text{Specification} \rightarrow \text{Context} \times \text{Result} \]

where

\[ \text{Specification} = \text{Input} \times \text{Precondition} \times \text{OutputVar} \times \text{Postcondition} \]

and \( \text{post}(\text{in}, \text{res}) \) holds for \( \text{pre}(\text{in}) \)

(i) expands knowledge \textit{within} the formal domain – “pure math”

(ii) expands knowledge \textit{outside} the formal model – “applied mat”
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Example: Calculation

Example for a problem to be solved by **Calculation**:

*in : function \( q_0, \text{length } L \)

*pre : \( q_0 \text{ is integrable in } x \land L > 0 \)

*out : function \( y(x) \)

*post : \( y(0) = 0 \land y'(0) = 0 \land V(0) = q_0.L \land M_b(L) = 0 \)

where \( V \) and \( M_b \) are constant function symbols in this theory of “bending lines”.

**Demonstration**
Check postcondition

Check if the Calculation has led to a correct result:

\[
\begin{align*}
y(x) &= \frac{q_0 \cdot L^2}{4 \cdot EI} \cdot x^2 - \frac{q_0 \cdot L}{6 \cdot EI} \cdot x^3 + \frac{q_0}{24 \cdot EI} \cdot x^4 \\
y'(x) &= \frac{q_0 \cdot L^2}{2 \cdot EI} \cdot x - \frac{q_0 \cdot L}{2 \cdot EI} \cdot x^2 + \frac{q_0}{6 \cdot EI} \cdot x^3 \\
y''(x) &= \frac{q_0 \cdot L^2}{2 \cdot EI} - \frac{q_0 \cdot L}{EI} \cdot x + \frac{q_0}{2 \cdot EI} \cdot x^2 = -\frac{M_b(x)}{EI} \\
M_b(x) &= -\frac{q_0 \cdot L^2}{2} + q_0 \cdot L \cdot x - \frac{q_0}{2} \cdot x^2 \\
M'_b(x) &= q_0 \cdot L - q_0 \cdot x = V(x) \\
V'(x) &= -q_0 = -q(x) \\
y(0) &= 0 \land y'(0) = 0 \land V(0) = q_0 \cdot L \land \\
\land M_b(L) &= -\frac{q_0 \cdot L^2}{2} + q_0 \cdot L \cdot L - \frac{q_0}{2} \cdot L^2 = 0
\end{align*}
\]
Example: Construction

Example for a problem to be solved by Construction:

\[
in : \text{points } \{A, B, C\}
\]

\[
pre : \neg(\{A, B, C\} \text{ are\_collinear})
\]

\[
out : \text{point } M
\]

\[
post : MA = MB \land MB = MC
\]

Demonstration
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Compare specifications

Calculations:

- **Specify** by translation of the problem into formal representation
- **Check precondition** $q_0 \text{ is integrable in } x \land L > 0$
- **Check result** i.e. calculate the postcondition $y(0) = 0 \land y'(0) = 0 \land V(0) = q_0 \cdot L \land M_b(L) = 0$

Constructions:

- **Specify** (partially) by input in geometric representation
- **Check precondition** $\neg(\{A, B, C\} \text{ are collinear})$
- **Check result**, i.e. the postcondition $\overline{MA} = \overline{MB} \land \overline{MB} = \overline{MC}$ by geometric means (length).

Both have the **same structure** in specifying and checking results.
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Three Kinds of Steps

A **step** propagates a Calculation/Construction by adding a new Term/geometric object (GeO)

\[
\text{step : } \text{Context} \times \text{State} \times \text{Pst} \times \text{Interact} \rightarrow \text{Context} \times \text{State} \times \text{Pst} \times \text{Term}
\]

\[
\text{step : } \text{Context} \times \text{State} \times \text{Pst} \times \text{Interact} \rightarrow \text{Context} \times \text{State} \times \text{Pst} \times \text{GeO}
\]

where

\[
\text{Pst} = \text{Program} \times (\text{Set of Locations})
\]

\[
\text{Interact} =
\begin{align*}
1. & \quad \text{Next} & \quad \text{Next} \\
2. & \quad \text{Tactic} & \quad \text{Tactic} \\
3. & \quad \text{Term} & \quad \text{GeO}
\end{align*}
\]

in Calculation in Construction
A **step** propagates a Calculation/Construction by adding a new Term/geometric object (GeO)

\[
\text{step} : \text{Context} \times \text{State} \times \text{Pst} \times \text{Interact} \rightarrow \text{Context} \times \text{State} \times \text{Pst} \times \text{Term} \\
\text{step} : \text{Context} \times \text{State} \times \text{Pst} \times \text{Interact} \rightarrow \text{Context} \times \text{State} \times \text{Pst} \times \text{GeO}
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1. & \quad \text{Next} \\
2. & \quad \text{Tactic} \\
3. & \quad \text{Term}
\end{align*}
\]

1. **Next**
2. **Tactic**
3. **Term**

in Calculation

------------

in Construction

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Models of Mathematics
The Program-interpreter of ISAC handles the three kinds of Interaction in Calculations:

<table>
<thead>
<tr>
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</tr>
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<tr>
<td>1. Next</td>
<td>execute Program in “debug-mode”, where the break-points are at Tactics . . .</td>
</tr>
<tr>
<td>2. Tactic</td>
<td>search for next Tactic in Program “equivalent” to the input Tactic . . .</td>
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<td>3. Term/GeO</td>
<td>check if Term/GeO can be generated within a “reasonable” scope in Program . . .</td>
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. . . such that the execution of Program can be resumed, and the programmer has not to deal with input and output.
The \textit{Program}-interpreter of \textit{ISAC} handles the three kinds of \textit{Interaction} in Calculations:

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### Tactics

**Demonstration**

<table>
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<tr>
<th>Tactics in Calculations</th>
<th>in (geometric) Constructions</th>
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<tbody>
<tr>
<td>rewrite : Term × Rule → Term</td>
<td>make_line : Point × Point → Line</td>
</tr>
<tr>
<td>rewrite_set : Term × Rule – set → Term</td>
<td>parallel : Line × Point → Line</td>
</tr>
<tr>
<td>compose : Term – set → Term</td>
<td>(... Euclid's 1st, 5th axiom)</td>
</tr>
<tr>
<td>split : Term → Term – set</td>
<td>? Tarski's axioms ?</td>
</tr>
<tr>
<td>subst : Term × Subst → Term</td>
<td>? Hilbert's axioms ?</td>
</tr>
<tr>
<td>subpbl : Specification → Term</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>intersect_1 : Circ × Circ → Point × Point</td>
</tr>
<tr>
<td></td>
<td>intersect_2 : Line × Line → Point</td>
</tr>
<tr>
<td></td>
<td>(...partial functions !?)</td>
</tr>
<tr>
<td></td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>subpbl : Specification → GeO</td>
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Program in ISAC

Demonstration

These programs would create/check the steps

```plaintext
program OrthoCenter P_1 P_2 P_3 =
M = let (l_1 = Bisector P_1 P_2 and l_2 = Bisector P_2 P_3)
    or (l_1 = Bisector P_1 P_2 and l_2 = Bisector P_1 P_3)
    or (l_1 = Bisector P_1 P_3 and l_2 = Bisector P_2 P_3)
in intersect_2 l_1 l_2

program Bisector P_1 P_2 =
l = let c_1 = Circle (P_1, Distance P_1 P_2);
c_2 = Circle (P_2, Distance P_1 P_2);
(Q_1, Q_2) = intersect_1 c_1 c_2
in make_line Q_1 Q_2
```

...without bothering the programmer with input and output (interpreters task).
Summary

Concluding requirements for educational math tools:

1. They shall be transparent and exhibit underlying logic
   - and gradually refine the logical rigor in educational use.

2. They shall be interactive models for stepwise doing math
   - and guide the student through the steps (Next | Tactic | GeO)

3. They shall converge to general tools (Algebra & Geom. . . )
   - and support changing representation for one and the same concept.